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# On the asymptotic behaviour of the Adem Thermodynamic Model for the ocean temperature prediction

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RESUMEN

Se analiza el comportamiento asintótico del modelo de Adem como una función del forzamiento externo y el coeficiente de disipación, se prueba la existencia de un conjunto atrayente acotado de las anomalías de la temperatura de la superficie del mar en una vecindad de cero. Se demuestra que la solución de la perturbación decae exponencialmente a cero, bajo una cierta condición. La razón de decaimiento depende del forzamiento externo y el coeficiente turbulento.

ABSTRACT

The asymptotic behaviour of Adem's model is analyzed as a function of the external forcing and the dissipation coefficient. The existence of a bounded attracting set of the sea surface temperature anomalies in a neighborhood of zero is proved. It is shown that the perturbation solution decays exponentially to zero, under a certain condition. The decay rate depends on the external forcing and the turbulent coefficient.

## 1. Introduction

During the last three decades a Northern Hemisphere Thermodynamic Climatic Model has been developed and applied to predict mean monthly anomalies of surface temperature (Adem, 1964, 1965, 1970; Adem, 1982; Adem, 1991).

The temperature anomaly predictions were verified over the contiguous U.S., showing a useful skill (Adem and Jacob, 1968; Adem and Donn, 1981; Donn *et al.*, 1985).

Predictions of the sea surface temperature anomalies and their month-to-month variation over the Atlantic and Pacific Oceans (Adem, 1970, 1975; Adem and Mendoza, 1988) have also shown a useful skill. In these experiments a simplified version of the model is used which includes only as a predicting equation the conservation law for the thermal energy applied to the upper layer of the oceans. This paper deals only with this ocean model.

Recent fluid dynamics research indicates that the asymptotical behaviour of a model, and a degree of similarity of the model attractor to that of the physical system is one of the important criteria to determine the quality of the model (Hale, 1988).

In this work, we investigate analytically the asymptotic behaviour of the Adem ocean temperature prediction model. The description of the model is given in section 2. In section 3, we show that under some assumptions on the structure of the external forcing, the temperature perturbation exponentially tends to zero. The rate of convergence to the attractor depends on the turbulent dissipation coefficient, geometry of the domain and the forcing structure (Skiba, 1990).

## 2. Description of the Prediction Model

The Adem Thermodynamic Model for the ocean is defined by a differential equation which represents the conservation of the thermal energy for the ocean upper layer (Adem, 1970):

$$\frac{\partial T_s}{\partial t} = -\vec{V}_{s_T} \cdot \nabla T_s + K_s \nabla^2 T_s + \frac{1}{h \rho_s C_s}(E_s - G_2 - G_3) - \frac{W}{h} \tag{1}$$

where $\nabla$ is the two-dimensional horizontal gradient operator; $T_s$, the surface ocean temperature; $\rho_s$, the constant density; $C_s$, the specific heat; $h$, the depth of the layer; $\vec{V}_{s_T}$, the normal horizontal velocity vector of the ocean current; $K_s$, a constant diffusion coefficient; $E_s$, the energy added by radiation; $G_2$ , the sensible heat given off to the atmosphere by vertical turbulent transport; $G_3$ , heat lost by evaporation and $W$, the heat transported down through the bottom of the layer.

It is assumed that the initial surface ocean temperature has a departure from normal, but that the atmospheric conditions, the ocean currents and the temperature at the bottom of the thermocline are normal (Adem, 1970). Observational studies and the numerical experiments carried out by Adem (1970, 1975) have shown that the prediction of the monthly or seasonal sea surface temperature anomalies depends mostly on the initial values of the anomalies themselves (see also Marchuk and Skiba, 1992), so that the other climate variables can be considered subordinated to the mixed layer temperature.

It is also assumed that the normal ocean temperature denoted by $T_{s_N}$ satisfies equation (1), provided the normal values of the parameters and heating functions are used.

Then following Adem (1971) and subtracting from (1) the corresponding equation for $T_{s_N}$, we obtain the equation

$$\frac{\partial T}{\partial t} = AD + TU + HE \tag{2}$$

for the ocean surface temperature anomaly $T = T_s - T_{s_N}$.

The terms $AD$, $TU$, and $HE$ are the parts of the temperature anomaly due to the horizontal transport of heat by the mean ocean currents (advection), the horizontal turbulent transport, and the total heating in the upper layer of the ocean, respectively:

$$AD = -\vec{V} \cdot \nabla T$$

$$TU = K \nabla^2 T$$

$$HE = -\beta_1 T \tag{3}$$

where $\vec{V}$ is the ocean current and $K$ is the turbulent coefficient in the model. In this work, for simplicity, the notation $\vec{V}$ and $K$ is used instead of $\vec{V}_{s_T}$ and $K_s$, respectively.

The term $HE$ is written as in Adem (1971), with

$$\beta_1 = 1.88 \times 10^{-6} \frac{| V_{A_N} |}{h} \tag{4}$$

where $| V_{A_N} |$ is the normal value of the surface wind speed.

Initial and boundary conditions are the following:

$$T(0,\ x,\ y) = T_o(x,\ y) \tag{5}$$

$$\left. \frac{\partial T}{\partial n} \right|_{\partial\Omega} = 0 \tag{6}$$

where $\partial\Omega$ is the boundary of the domain $\Omega$ and is the temperature anomaly derivative in the direction of the normal vector $\vec{n}$ to the boundary (Fig. 1).

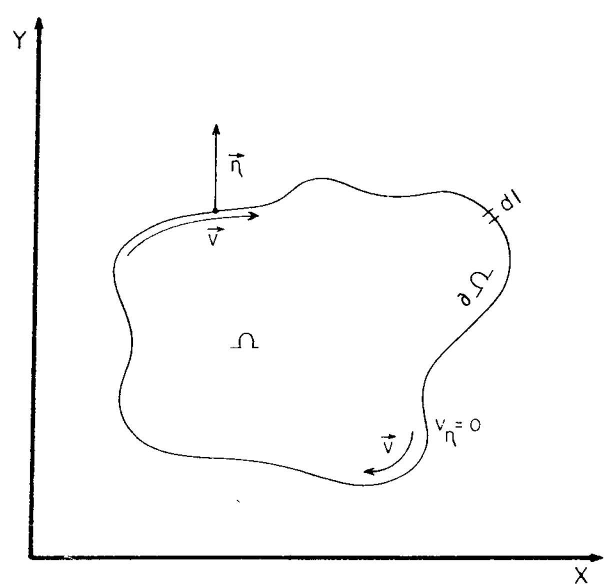


Fig. 1. Integration Domain. Here $\vec{n}$ denotes the outward normal vector at the boundary $\partial\Omega$ of $\Omega$, $V_n$ is the normal component of the ocean current on the boundary, and $d\iota$ is the infinitesimal element of length along the boundary line $\partial\Omega$.

We assume that the normal component of the ocean current is zero on the boundary:

$$\left. V_n \right|_{\partial\Omega} = 0 \tag{7}$$

and the flow is incompressible:

$$\text{div}\ \vec{V} = 0 \tag{8}$$

Due to conditions (6), (7), and (8), the temperature anomalies $T(t,\ x,\ y)$ satisfy the equation

$$\frac{\partial}{\partial t} \int_{\Omega} T(t,\ x,\ y) d\Omega + \beta_1 \int_{\Omega} T(t,\ x,\ y) d\Omega = 0$$

and hence,

$$\int_\Omega T(t,\ x,\ y)d\Omega = \{\int_\Omega T_o(x,\ y)d\Omega\}\exp(-\beta_1 t)$$

Thus, if $\int_\Omega T_o(x,\ y)d\Omega = 0$ at the initial moment then

$$\int_\Omega T(t,\ x,\ y)d\Omega = 0 \tag{9}$$

for all times.

In this work we consider only such perturbations which satisfy the condition (9). It is equivalent to the assumption that the average value over the whole domain $\Omega$ of the monthly mean temperature anomaly is zero. This assumption is physically not so restrictive if $\Omega$ is the whole ocean.

## 3. Study of the Asymptotic Behaviour of the Thermodynamic Model

We assume that $K$ and $\beta_1$ are constant parameters in equation (2).

Let $\Omega$ be the domain of integration (Fig. 1), and $L^2(\Omega)$ be the Hilbert space of square integrable functions on $\Omega$, with the scalar product

$$< T,\ g >= \int_\Omega T g d\Omega \tag{10}$$

where $T(x,\ y)$ and $g(x,\ y)$ are two arbitrary functions defined in the domain $\Omega$ and the norm

$$||T|| =< T,\ T >^{1/2} \tag{11}$$

where

$$d\Omega = dxdy,\ (x,\ y)\epsilon\Omega.$$

The scalar product of equation (2) with T gives

$$< \frac{\partial T}{\partial t}, T >=< AD,\ T > + < TU,\ T > + < HE,\ T > \tag{12}$$

where from (3)

$$< HE,\ T >= -\beta_1 < T,\ T >= -\beta_2||T|| \tag{13}$$

Using the boundary condition (6) we obtain

$$< TU,\ T >= K < \nabla^2 T,\ T >= -K||\nabla T||^2 \tag{14}$$

for the turbulent diffusion term.

For the term of advection by ocean currents we have

$$< AD,\ T >= - < \vec{V}\cdot\nabla T,\ T >= -\frac{1}{2}\int_\Omega \vec{V}\cdot\nabla(T^2)d\Omega \tag{15}$$

Taking into account the continuity equation (8) and Green's formula (Sokolnikoff, 1958), equation (15) can be written as

$$< AD,\ T >= -\frac{1}{2}\int_{\Omega} \nabla \cdot (\vec{V}T^2)d\Omega = -\frac{1}{2}\oint_{\partial\Omega} V_n T^2 d\iota$$

where $\oint$ is the curvilinear integral over $\partial\Omega$ and $d\iota$ is the infinitesimal element of length along the boundary line $\partial\Omega$ (Fig. 1). Thus the boundary condition (7) leads to

$$< AD,\ T >= 0 \tag{16}$$

Finally, for the storage of thermal energy we obtain

$$< \frac{\partial T}{\partial t}, T >= \frac{1}{2}\frac{\partial}{\partial t}||t||^2 = ||T||\frac{\partial}{\partial t}||T|| \tag{17}$$

Substituting (13), (14), (16) and (17) in (12) results in

$$\frac{1\partial}{2\partial t}||T||^2 = -K||\nabla T||^2 - \beta_1||T||^2 \tag{18}$$

We now use Poincare inequality (Chipot, 1984)

$$||T|| \leq C||\nabla T|| \tag{19}$$

with the constant $C > 0$ that depends only on the geometry of the domain $\Omega$. The constant $C$ is positive in our case, because of the assumption (9) for the temperature anomalies T (see Appendix A). Then

$$-\frac{K}{C^2}||T||^2 \geq -K||\nabla t||^2$$

Substituting the last inequality in equation (18) we obtain

$$\frac{1\partial}{2\partial t}||T||^2 \leq -\frac{K}{C^2}||T||^2 - \beta_1||T||^2$$

or

$$||T||\frac{\partial}{\partial t}||T|| \leq -(\beta_1 + \frac{K}{C^2})||T||^2$$

Dividing by $||T||$ we obtain

$$\frac{\partial}{\partial t}||T|| + (\beta_1 + \frac{K}{C^2})||T|| \leq 0 \tag{20}$$

Long-time behaviour of solutions $T(t,\ x,\ y)$ will be analyzed now. Multiplying the inequality (20) by

$$\exp(\beta_1 + \frac{K}{C^2})t$$

leads to

$$\frac{\partial}{\partial t}[||T|| \exp(\beta_1 + \frac{K}{C^2})t] \leq 0 \tag{21}$$

Integrating (21) over time from $t = 0$ up to a moment $t$ results in

$$||T(t,\ x,\ y)|| \leq ||T_o(x,\ y)|| \exp[-(\beta_1 + \frac{K}{C^2})t] \tag{22}$$

Since $\beta_1$ is positive (see (4)), the zero solution is the only attractor in our system:

$$T(t,\ x,\ y) \to 0 \quad \text{as} \quad t \to \infty$$

As $T(t) = T_s(t) - T_{s_N}(t)$, it means that

$$\lim_{t\to\infty} ||T_s(t) - T_{s_N}(t)|| = 0 \tag{23}$$

Thus, the ocean temperature $T_s(t)$ will eventually tend to the normal climatic value. Schematically it can be represented as in Figure 2.

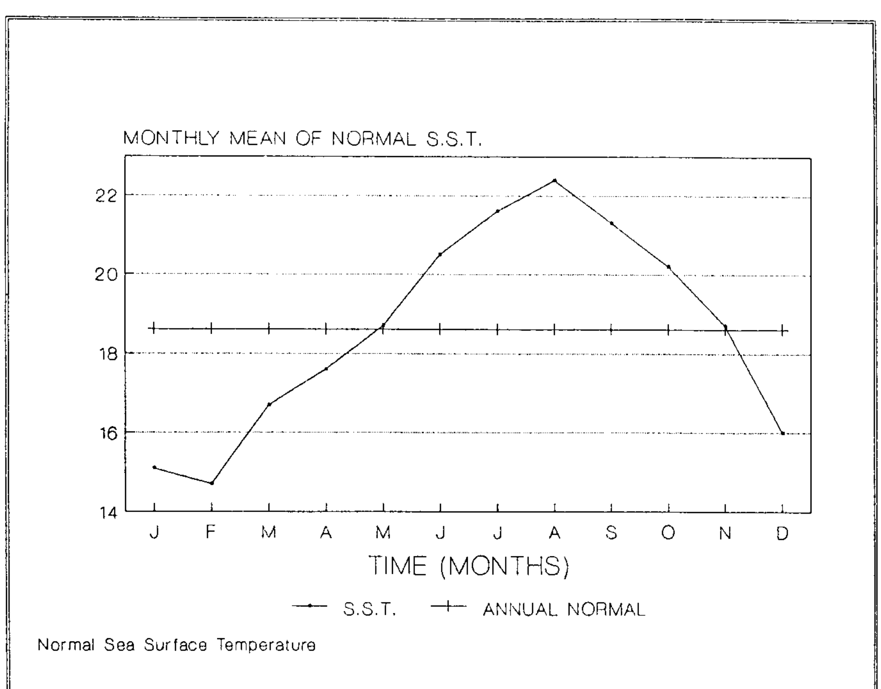


Fig. 2. Evolution of the anomaly of the surface ocean temperature, according to (23).

## 4. Conclusions

It is shown that the zero solution is the only attractor of equation (2), i.e., any solution $T(t, x, y)$ of the equation (2) eventually tends to zero. It means that the ocean temperature $T_s(t)$ (see Eq.(1)) approaches the normal climatic value $T_{s_N}(t)$.

Thus, any bounded neighborhood of the zero is an absorbing set for the equation (2) solutions.

Note, that the smaller are $\beta$ and $K$, the slower is the rate of convergence of all solutions $T(t, x, y)$ to zero and vice versa. Besides, as it follows from (22), the rate of decreasing also depends on the constant $C$ of the Poincaré inequality (19).

As a simple example, suppose that our domain $\Omega$ coincides with the whole Northern Hemisphere, the parameter $\beta_1$ is defined by (4), and the diffusion coefficient $K$ is equal to $3 \times 10^8$ cm$^2$ sec$^{-1}$.

We obtain $\frac{|V_{A_N}|}{h} = 10^{-1}$ sec$^{-1}$, considering the pairs of values: with $| V_{A_N} |= 5$ m sec$^{-1}$ and $h = 50$ m or $| V_{A_N} |= 10$ msec$^{-1}$ and $h = 100$ m (Adem, 1971).

In this case, the eigenvalues of Laplace operator on the hemisphere are equal to $N(N+1)/R^2$ where $N$ is natural and $R = 6.37 \times 10^6$ m is the radius of the sphere. Since $1/C^2$ is equal to the lowest nonzero eigenvalue, $1/C^2 = 2/R^2$ . Then this leads to a factor $K/C^2 = 0.01478 \times 10^{-7}$ sec$^1$.

As a result, $\beta_1 + \frac{K}{C^2} = 1,894 \times 10^{-7}$ sec$^{-1} = \tau$.

Therefore the e-folding time of decreasing the norm (22) of the perturbations is equal to $\frac{1}{\tau} = 60.9$ days $\sim 2$ months.

We now make an important note.

We have analyzed the behaviour of the perturbation Eq. (2). The term $HE$ in this equation is a result of the linearization of the forcing (see Appendix B)

$$f(T_s) = \frac{1}{h\rho_s C_s}(E_s - G_2 - G_3) - \frac{W}{h} \tag{24}$$

Exponential decreasing of any small perturbations obtained in the previous section (see Eq. (22)) has been shown for the particular case $\beta_1 =$ constant $> 0$ that corresponds to a linearized heating term. In the non-linearized case, $\beta_1$ is not constant, and perturbations will exponentially tend to zero if

$$\frac{K}{C^2} > \max_{x,\ y, \epsilon\Omega} \max_{t} | \beta_1(t, x, y) | \tag{25}$$

The condition (25) also takes into account the case of a stationary, periodic and bounded (in time) non-stationary normal solution $T_{s_N}(t, x, y)$ (see appendix B).

For example, since the behaviour of the normal monthly mean values $T_{s_N}(t)$ is approximately periodic during the year, the solution $T_s(t)$ of Eq.(1), will tend to the climatic cycle independently of the initial condition $T_s(0, x, y)$, (Fig. 3).

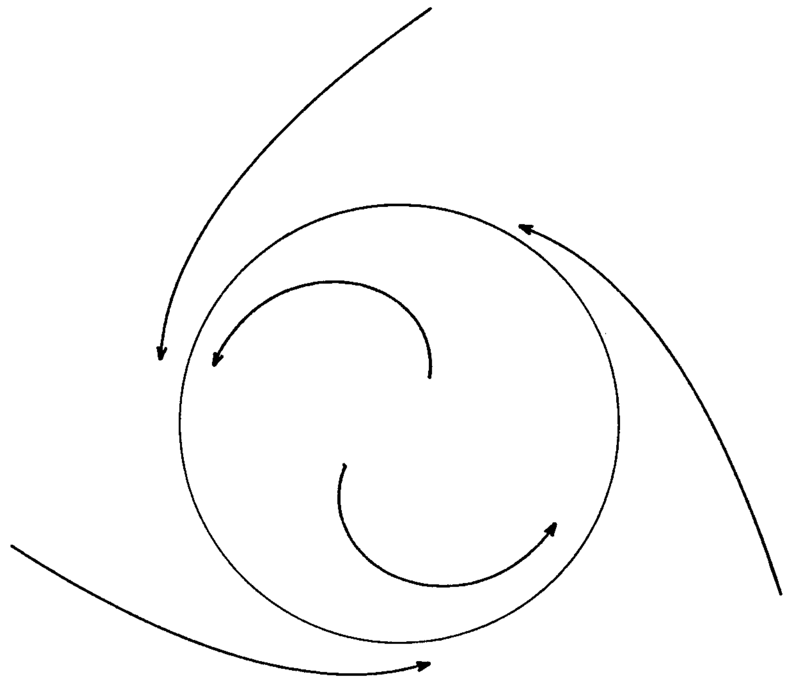

Fig. 3. Evolutions with a cycle attractor of the surface ocean temperature anomaly. All the trajectories located inside the limit cycle tend to move asymptotically to it, just as the trajectories on the outside drift inward to the limit cycle.

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## APPENDIX A

### THE POINCARE INEQUALITY

Due to Eq. (14), we obtain

$$< -\nabla^2 T,\ T >= \|\nabla T\|^2 \geq O \tag{A.1}$$

for any function $T$. It means that $-\nabla^2$ is nonnegative operator.

The spectral problem

$$-\nabla^2\varphi_n = \omega_n\varphi_n$$

$$\frac{\partial\varphi_n}{\partial_n}|_{\partial\Omega} = 0 \qquad (A.2)$$

gives us an orthonormal basis of the real functions $\{\varphi_n(x,\ y)\}$.

Thus

$$T = \sum_n T_n\varphi_n(x,\ y) \qquad (A.3)$$

for any function $T$ where

$$T_n =< T,\ \varphi_n >$$

is the Fourier coefficient.

Suppose that the eigenvalue problem (A.2) is solved, and the basic functions $\varphi_n(x,\ y)$ are known. Let us substitute the series (A.3) in (A.1). Then

$$||\nabla T||^2 = \sum_n T_n \sum_m T_m < -\nabla^2\varphi_n,\ \varphi_m >$$

Using (A.2), we obtain

$$||\nabla T||^2 = \sum_n \omega_n T_n \sum_m T_m < \varphi_n,\ \varphi_m >$$

Since the functions $\varphi_n$ are orthonormal, i.e.,

$$< \varphi_n,\ \varphi_m >= \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

where $\delta_{mn}$ is called the Kronecker delta (Dennery and Krzywicki, 1967), we obtain

$$||\nabla T||^2 = \sum_n \omega_n T_n^2 \qquad (A.4)$$

It follows from (A.1) and (A.4) that $\omega_n \geq 0$.

Let us numerate all the eigenvalues in the order of increasing their values, i.e., $\omega_n < \omega_{n+1}$, $n = 1,\ 2,\ 3, \ldots$

We will show now that in fact all $\omega_n$ are positive under the condition (9), and hence,

$$min\{\omega_n\} = \omega_1 > 0$$

Indeed, for the spectral problem (A.2), the eigenvalue $\omega$ is zero only if

$$\varphi_n(x,\ y) = \text{constant}.$$

But, due to Eq.(9), such constant functions are excluded from our consideration. Actually, if

$$\int_\Omega T d\Omega = 0$$

with $T$ = constant we have

$$T \int_\Omega d\Omega = 0$$

where $\int_\Omega d\Omega$ is not zero, so that $T$ = constant = 0

As a result,

$$\|\nabla T\|^2 = \sum_n \omega_n T_n^2 > \omega_1 \sum_n T_n^2 \qquad (A.5)$$

for any function $T$.

Taking into account that

$$\sum_n T_n^2 =< T,\ T >= \|T\|^2$$

the equation (A.5) can be written as

$$\|\nabla T\|^2 \geq \omega_1 \|T\|^2$$

or

$$\|\nabla T\| \geq \sqrt{\omega_1} \|T\|$$

We now can obtain the Poincaré inequality (19)

$$\|T\| \leq \frac{1}{\sqrt{\omega_1}} \|\nabla T\| \qquad (A.6)$$

with

$$C = \frac{1}{\sqrt{\omega_1}}$$

## APPENDIX B

### THE FORCING STRUCTURE

The equation (1) can be written as

$$\frac{\partial T_s}{\partial t} = -V_{s_T} \cdot \nabla T_s + K \nabla^2 T_s + f(T_s) \qquad (B.1)$$

Substracting from (B.1) the corresponding equation for $T_{s_N}$,

$$\frac{\partial T_{s_N}}{\partial t} = -V_{s_T} \cdot \nabla t_{s_N} + K\nabla^2 T_{s_n} + f(T_{s_N}) \tag{B.2}$$

we obtain

$$\frac{\partial T}{\partial t} = -V_{s_T} \cdot \nabla T + K\nabla^2 T + f(T_s) - f(T_{s_N}) \tag{B.3}$$

for the ocean surface temperature anomaly $T = T_s - T_{s_N}$.

In the linearized heating function that we have used, we have assumed that

$$T_s = T_{s_o} + T'_s \qquad T_{s_o} >> T'_s$$

$$T_{s_N} = T_{s_o} + T'_{s_N} \qquad T_{s_o} >> T'_{s_N}$$

where $T_{s_o}$ is the constant mean annual surface temperature.

Therefore,

$$f(T_s) = f(T_{s_o}) + \frac{\partial f(T_{s_o})}{\partial T_s} T'_s \tag{B.4}$$

$$f(T_{s_N}) = f(T_{s_o}) + \frac{\partial f(T_{s_o})}{\partial T_s} T'_{s_N} \tag{B.5}$$

Subtracting (B.5) from (B.4), we obtain

$$f(T_s) - f(T_{s_N}) = -\beta_1 T$$

where

$$\beta_1 = -\frac{\partial f(T_{s_o})}{\partial T_s}$$

The previous analysis has been performed for the linearized case $f(T_s) - f(T_{s_n}) = -\beta_1 T$, where $\beta_1$ is a constant. But for the non-linearized case $f(T_s) - f(T_{s_n})$ has a more complicated form.

In this case, let us suppose that $||T||$ is sufficiently small so that we can use Newton series truncation by the linear term

$$f(T_s) = f(T_{s_N}) + \frac{\partial f(T_{s_N})}{\partial t_s} T + O(T^2)$$

It means that

$$f(T_s) - f(T_{s_N}) \sim -\beta_1 T \tag{B.6}$$

where

$$\beta_1 = -\frac{\partial f(T_{s_N})}{\partial t_s}$$

and

$$T_{s_N} = T_{s_N}(t,\ x,\ y)$$

In this case, in which $\beta_1$ is not constant, the condition (26) is the sufficient one for the exponential decrease of small perturbations $T(t,\ x,\ y)$.

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