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The zonal atmospheric structure: A heuristic theory

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RESUMEN

Se formula una teoría heurística de la estructura zonalmente promediada sobre una Tierra esférica empleando una parametrización de la influencia de los torbellinos sobre la estructura zonal. Las influencias principales de éstos son los transportes meridionales de calor y momento. Dichos transportes se expresan en función de cantidades promediadas zonalmente mediante el uso de la suposición que los transportes de cantidades casi conservativas pueden parametrizarse por un proceso de difusión con un valor constante del coeficiente. El principio es empleado en los tranportes de vorticidad y temperatura potenciales, de donde se muestra que el transporte de momento puede obtenerse indirectamente.

La teoría se formula para el modelo casi geostrófico de dos niveles con calor y disipación, mediante el desarrollo de las variables dependientes zonalmente promediadas en polinomios de Legendre. Las ecuaciones casi geostróficas suplementadas por una ecuación lineal de balance son suficientes para satisfacer las relaciones de paridad válidas sobre la esfera.

Las aplicaciones de la toría son llevadas a cabo empleando varios calentamientos zonalmente promediados basados en cómputos de observaciones o en simples especificaciones. En el primer caso se muestra que la teoría, en un modo cualitativamente correcto, puede explicar los perfiles meridionales de los vientos zonales en los diversos niveles, para la circulación meridional media y para los transportes de momento deducidos a partir del transporte meridional de vorticidad.

Las soluciones para los diversos valores del coeficiente que determina el transporte meridional de la vorticidad potencial muestra que la conversión de energía proveniente de los torbellinos en energía cinética zonal es particularmente sensible a la intensidad del transporte de vorticidad potencial. Si el transporte disminuye resulta que la conversión se vuelve negativa de tal manera que la conversión prosigue desde la energía zonal hacia la energía cinética torbellinaria. Se sabe que esta situación corresponde a los casos de bloqueo intenso.

Una simple especificación del calentamiento, simulando calentamiento en las latitudes bajas y enfriamiento en las altas latitudes es insuficiente para dar una imagen cualitativamente correcta de la circulación meridional media.

ABSTRACT

A heuristic theory of the zonally averaged structure of the atmosphere is formulated on the spherical Earth using parameterization of the influence of the eddies on the zonal structure. The principal influences of the eddies are the meridional transports of heat and momentum. These transports are expressed in terms of zonally averaged quantities by using the assumption that the transports of quasi-conservative quantities may be parameterized by a diffusion process with a constant value of the coefficient. The principle is used on the transports of potential vorticity and of potential temperature, whereafter it is shown that the momentum transport may be obtained indirectly.

The theory is formulated for the quasi-geostrophic two-level model with heating and dissipation using an expansion of the dependent zonally averaged variables in Legendre polynomials. The quasi-geostrophic equations supplemented by a linear balance equation are sufficient to satisfy the parity relations valid on the sphere.

Applications of the theory are carried out using zonally averaged heating based on calculations from observations or on simple specifications. In the first case it is shown that the theory, in a qualitatively correct way, can account for the meridional profiles of the zonal winds at the various levels, for the mean meridional circulation and for the meridional transports of momentum derived from the meridional transport of vorticity.

Solutions for various values of the coefficient determining the transport of potential vorticity show that the energy conversion from eddy to zonal kinetic energy is particularly sensitive to the strength of the potential vorticity transport. If the transport is decreased, it turns out that the conversion becomes negative in such a way that the conversion goes from the zonal to the eddy kinetic energy. It is known that this situation corresponds to intense cases of blocking.

A simple specification of the heating, simulating heating in the lower and cooling in the higher latitudes is insufficient to give a qualitatively correct picture of the mean meridional circulation.

1. Introduction

A classical problem in meteorology is the explanations of the zonal structure of the atmosphere. It is well known that Halley (1686) and Hadley (1735) were among the first to speculate on why the surface zonal winds had the observed distributions. Later modifications to the original schemes were done by a number of scientists, but Ferrel (1856) is especially remembered because his name is attached to the indirect middle latitude meridional cell. An account of the classical speculations on the subject is given by Lorenz (1967).

The modern view puts major emphasis on the rôle of the eddies. Observational studies and numerical integrations of equations describing the general circulation of the atmosphere have shown that the zonal structure of the atmosphere can be described in terms of the diabatic heating, the meridional transport of heat and momentum and the frictional dissipation at the surface of the Earth and in the interior of the atmosphere. Theoretical studies such as the investigation conducted by Charney (1959) describes the zonal structure in a schematical way by computing the zonal structure as a result of the diabatic heating and the frictional dissipation as a first step. It is then shown that the calculated structure is unstable for small perturbations. Using an *ad hoc* assumption concerning the preservation of shape of the infinitesimal growing wave and permitting only the nonlinear interaction between that wave and the zonally averaged quantities it is possible to determine the steady state amplitude of the wave and then determine the transports of heat and momentum by the wave. In the theory it has thus been assumed that the nonlinear cascade of energy by the waves is negligible compared to the interaction between the wave and the zonal structure. The theory cannot therefore account for the spectral distribution of energy on the various wave numbers. It is also unable to pay attention to the difference between forced and free waves.

A quite different, but not necessarily more complete point of view has been used primarily by Green (1968, 1970) and Wiin-Nielsen (1968, 1971). These considerations amount to a parameterization of the influence of the eddies on the zonal flow. While such a point of view was advanced already by Defant (1921) who described the transport of sensible heat as a diffusion process, it has been difficult to expand the concept to other transport processes in the atmosphere. Some of the first studies of the eddy momentum transport in the atmosphere by Starr and White (1951) and Buch (1954) indicated clearly that a simple diffusion process cannot be used to describe the influence of the eddy momentum transport on the zonal flow. A guiding principle in the studies by Green (*loc. cit.*) and Wiin-Nielsen (*loc. cit.*) is that only quasi-conservative quantities may be parameterized as diffusion processes. Such a principle is of course in agreement with the general empirical description of a diffusion process since it rests on the assumption of conservation. The concept may therefore be used on the transport of heat where the quasi-conservative quantity is the potential temperature, and on the meridional transport of potential vorticity, where the potential vorticity is conserved in the large-scale adiabatic and frictionless flow. As it turns out, the meridional momentum transport become a secondary quantity that may be calculated from the parameterizations. These ideas have been tested in a number of observational and theoretical studies by Wiin-Nielsen and Sela (1971), Wiin-Nielsen and Fuenzalida (1975), White and Green (1982, 1984) and by Wu and White (1986).

The approach described above can accept a spectrum of waves, but it rests on the assumption that the major wave groups in the spectrum transport the conservative property in qualitatively the same way. In particular, it may be applied because the forced and the free waves obey the assumption as shown from observations by Wiin-Nielsen *et al.* (1963, 1964). In its most general form the procedure is nonlinear, because the diffusion factors depend on the flow, and all but one are determined from the integral constraints that apply to the flow. Such an application has been made by Wiin-Nielsen (1988), but the study was limited to only a few components. i.e. to a low order system. The main idea has its limitations. While it may be applicable to the general averaged atmospheric flow, it is probably unable to describe extreme flow configurations such as blocking situations. More important is the weakness that while it rests on some general properties of the structure of the atmospheric waves, it is unable to say anything at all about the development and changes of the waves themselves.

The purpose of the present investigation is to describe some of the major features of the atmospheric general circulation using a model of the above type and keeping the procedures as simple as possible. In such a case one cannot expect that one can account for all the details, but only that the results will be qualitatively correct with respect to the general structure of the derived quantities and their order of magnitude. We shall in particular see how far we may proceed using constant values of the diffusion coefficients. Furthermore, we shall formulate the problem on the sphere and use an expansion of the scalar quantities in series of Legendre functions of the first kind. Since we shall deal with zonal averages only, it will be sufficient to use the Legendre polynomials. The study will make extensive use of the properties of these functions, and the reader is refered to standard texts such as Jahnke, Emde and Lösch (1960), Abramowitz and Stegun (1972) or similar collections for the details of the various formulas used in the developments and derivations.

The model used in the study is the standard two-level, quasi-nondivergent model. The vertically integrated heating entering the model will be specified in advance. The study will be limited to the case where the heating is symmetrical with respect to the equator. This assumption implies that the geopotential and the temperature have the same symmetry, while quantities such as the streamfunction and the vorticity are asymmetric around the same curve. It will thus be necessary to formulate the model in such a way that these properties are used correctly. For this purpose we employ a linear form of the balance equation. With the full variation of the Coriolis parameter in the balance equation we can satisfy the requirements. The use of the linear balance equation is not without problems as was pointed out by Eliasen and Machenhauer (1965). It turns out that it may be replaced by an infinite set of relations permitting the calculation of a component of the streamfunction, say number n, from the component (n-1) in the geopotential and the component (n-2) in the streamfunction provided that the first component may be obtained by a special procedure. This difficulty is circumvented in observational studies by relating the first component of the streamfunction to the total angular momentum of the zonal flow. This procedure cannot be used directly in the present study because the zonal flow is unknown. However, we can assume that the integrated angular momentum approximately vanishes since we are computing the steady state flow created by the prescribed heating. One may therefore imagine that the angular momentum is zero at the start of the heating process, and that it remains so during the approach to the steady state. In addition, one can show that the first component is zero for the geostrophic flow with a full variation of the Coriolis parameter as employed elsewhere in the study. Due to the nature of the linear balance equation it is a necessity to select a reasonable maximum value of the components included in the calculations. The maximum value of the number of Legendre functions is determined by restricting the number of components to those for which the basic assumptions of quasi-balanced flow is satisfied.

The results of the study will show that for a realistic heating function we may obtain accept-able values of the zonal winds, the transports of heat, potential vorticity and, indirectly, of momentum, and we are thus able to conclude that the theory is capable of accounting for some gross aspects of the general atmospheric circulation.

2. Parameterizations

The purpose of the parameterizations is to relate various meridional transports to the zonal mean quantities. As mentioned briefly in the introduction it is done by applying the diffusion approximation to quasi-conservative parameters such as the potential vorticity and the potential temperature. The adopted model will be a standard two-level, quasi-nondivergent model driven by the zonally averaged heating and incorporating standard dissipation mechanisms.

In a two-level, quasi-nondivergent model the streamfunction and the quantities related to it are carried at an upper (250 hPa) and a lower (750 hPa) level. These levels are indicated by subscripts 1 and 3, respectively. The conserved quantity at level 1 is:

$$\xi_1 = \varsigma_1 - q^2 \Psi_T \tag{1.1}$$

where ξ_1 is the potential vorticity, while ς_1 is the ordinary vorticity. The remaining symbols are defined as follows:

$$q^2 = \frac{2\Omega^2}{\sigma p^2}$$

$$\Psi_T = (\Psi_1 - \Psi_3)/2 \tag{1.2}$$

where σ is the usual measure of the static stability in the p-system, Ω is a standard value of the Coriolis' parameter, and $P = 500$ hPa.

The main assumptions as described in the introduction are the diffusion assumptions for the potential vorticity transport and the heat transport. They are:

$$(\xi_1 v_1)_z = -K_1 \frac{\partial \xi_{1z}}{a \partial \varphi}$$

$$(\Psi_T v_1)_z = -K_H \frac{\partial \Psi_{Tz}}{a \partial \varphi} \tag{1.3}$$

in which the K's are the diffusion coefficients, a the radius of the Earth and φ is the latitude.

Using the formulas given above one can obtain the vorticity transport from its definition and the transport parameterizations. Denoting the meridional vorticity transport by V_1 it turns out to be most convenient to multiply by cosinus to latitude in which case one gets after rearrangement:

$$V_1 \cos\varphi = -\frac{K_1}{a}(1-\mu^2)\frac{\partial \varsigma_{1z}}{\partial \mu} - \frac{q^2}{a}(K_H - K_1)(1-\mu^2)\frac{\partial \Psi_{Tz}}{\partial \mu} \tag{1.4}$$

To ease the calculations we shall nondimensionalize all scalar quantities by using a length scale measured by the radius of the Earth and a timescale defined by the inverse of the angular velocity of the Earth. In addition we write all the scalar variables as series of Legendre polynomials. Using the properties of these functions and denoting all nondimensional quantities by a 'hat' we obtain:

$$\hat{V}_1 \cos(\varphi) = -\sum[\hat{K}_1\hat{\varsigma}_1(n) - \lambda^2(\hat{K}_H - \hat{K}_1)\frac{\hat{\varsigma}_T(n)}{n(n+1)}](1-\mu^2)\frac{dP_n}{d\mu} \tag{1.5}$$

where the summation in all cases is carried out over the number n. We note that $\lambda^2 = a^2q^2$ and is a nondimensional quantity.

A formula for the derivative of a Legendre polynomial is:

$$(1-\mu^2)\frac{dP_n}{d\mu} = \frac{n(n+1)}{2n+1}(P_{n-1} - P_{n+1}) \tag{1.6}$$

This expression may be inserted in (1.5), and it is the possible to sum the series and to calculate the vorticity transport at level 1. Although a division by cosinus is necessary there will be no problems at the poles because formula (1.6) shows that the right hand side of (1.5) is divisible by the cosinus function. This may be seen by the argument that the derivative of the Legendre polynomial is itself some polynomial, and (1.6) shows then that the difference between the two Legendre polynomials of order $(n-1)$ and $(n+1)$ has $(1-\mu^2)$ as a factor. We may use an analogous procedure to derive a formula for the vorticity transport at level 3 where the conservative quantity is:

$$\xi_3 = \varsigma_3 + q^2\Psi_T \tag{1.7}$$

The final formula corresponding to (1.5) may be written as follows:

$$V_3 \cos(\varphi) = -\sum[\hat{K}_3\hat{\varsigma}_3(n)n(n+1) - \lambda^2(\hat{K}_3 - \hat{K}_H)\hat{\varsigma}_T(n)]\frac{P_{n-1} - P_{n+1}}{2n+1} \tag{1.8}$$

In the following we shall use (1.5) and (1.8) to obtain the momentum transports at levels 1 and 3. For this purpose we introduce the notations:

$$m_1 = (u_1v_1)_z$$

$$m_3 = (u_3v_3)_z \tag{1.9}$$

Using the well known formula:

$$V_1 = \frac{\partial m_1 \cos^2(\varphi)}{a\cos^2(\varphi)\partial\varphi} \tag{1.10}$$

and the corresponding formula for the momentum transport at level 3 we may derive the nondimensional expression:

$$\hat{V}_1 \cos(\varphi) = -\frac{\partial(\hat{m}_1 \cos^2(\varphi))}{\partial\mu} \tag{1.11}$$

from which the momentum transport at level 1 may be obtained by integration from an arbitrary value of μ to 1 i.e. the North Pole. Also in this case there will be no problems at the Poles since the left hand side is divisible by $(1-\mu^2)$ after integration using an argument analogous to the one applied above. It is seen that the same procedure may be applied to obtain the momentum transport at level 3.

Next we turn the attention to the heat transport. The general definition of the heat transport across a latitude circle is:

$$H_t = \frac{p_o}{g} \int_o^1 \int_o^{2\pi} c_p T v a \cos(\varphi) d\lambda dp_* \tag{1.12}$$

where g is the acceleration of gravity, p_o a standard value of the surface pressure and $p_* = p/p_o$. Using the gas equation and the hydrostatic equation to replace T by the thermal streamfunction ψ_T we may for the two-level model write:

$$H_t = 4\pi a \frac{p_o}{g} \frac{c_p}{R} \Omega (\Psi_T v)_z \cos(\varphi) \tag{1.13}$$

Going to a nondimensional form and introducing once again the series expansion in Legendre polynomials we obtain finally:

$$H_t = c_h \hat{K}_H \sum \frac{\hat{\varsigma}_T(n)}{2n+1} (P_{n-1} - P_{n+1})$$

where

$$c_h = 4\pi \frac{p_o}{g} \frac{c_p}{R} a^4 \Omega^3 \tag{1.15}$$

In this formulation the heat transport will be expressed in energy per unit time, i.e. in $J_{s^{-1}}$.

This concludes the formulation of the parameterizations where we have expressed the transport processes in terms of zonally averaged quantities. It will be seen that we have selected to use the Legendre coefficients for vorticity and not for the streamfunction. This choice is made because it simplifies the expressions. Whenever a Legendre coefficient in the streamfunction appears it has been replaced by the corresponding coefficient for the vorticity using the relationship:

$$\hat{\varsigma}(n) = -n(n+1)\Psi(n) \tag{1.16}$$

In this section we have formulated the parameterizations in the most simple form by assuming that the values of the diffusion coefficients are constants. The assumption is not realistic because observational studies have shown that the diffusion coefficients have a tendency to be at a maximum in middle latitudes with smaller values closer to the Poles and the equator. It is possible to include such a fixed variation in the model as demonstrated by Wiin-Nielsen (1988) where it was assumed that the latitudinal variation of the diffusion coefficients were given expressions such as

$$K = K_m \mu^2 (1-\mu^2) \tag{1.17}$$

but, while (1.17) may be a reasonable approximation to an averaged variation of the diffusion coefficient, it is more likely that the dependence on latitude is determined by the nature of the zonally averaged quantities, and such a dependence is much more difficult to formulate.

3. Stationary States

Using the parameterizations developed in Section 2 it is the purpose of this section to give a description of the calculation of the stationary states of the two-level model. The formulation of the calculation requires some care due to parity relations existing among the various parameters. As an example we may consider a heating that is symmetrical around the equator. In such a case the thermodynamic equation dictates that the geopotential and the vertical velocity are also symmetrical, while the streamfunctions and the vorticities are asymmetrical. It follows then that the zonal winds are symmetrical, while the eddy transports become asymmetrical.

The standard two-level, quasi-nondivergent model is also quasi-geostrophic because the streamfunction is obtained by a rescaling of the geopotential by dividing by a standard value of the Coriolis parameter. It is therefore not applicable to our problem because the streamfunction and the geopotential are of the same parity. It is therefore necessary to generalize the model slightly. This will in this study be done by employing a simplified form of the balance equation that expresses a balance between the pressure force and the Coriolis force, where the latter force is expressed by the nondivergent wind. The balance equation becomes in this case:

$$f\nabla^2\Psi + \nabla f \cdot \nabla\Psi = \nabla^2\phi \tag{3.1}$$

Equation (3.1) is brought into nondimensional form, reduced to the meridional components, and the Legendre series are introduced. Using a number of formulas for the Legendre polynomials and the orthogonality properties one may transform (3.1) to the following recursion formula:

$$2\frac{n}{2n+3}\hat{\varsigma}(n+1) + 2\frac{(n+2)}{2n-1}\hat{\varsigma}(n-1) = \hat{\phi}(n) \tag{3.2}$$

(3.2) will in general permit the calculation of component $(n+1)$ of the vorticity from component n of the geopotential and component $(n-1)$ of the vorticity. However, it is seen that the recursion relation does not permit the calculation of the first component of the vorticity. This should be done by setting $n = 0$, but in that case we find that the factor in front of the first component is zero. It is therefore necessary to invent a special starting procedure. In observational studies, where the difficulty has been encountered before (Eliasen and Machenhauer, 1965) it can be circumvented by noting that

$$\hat{\varsigma}(1) = \int_{-1}^{1} \mu\hat{\varsigma}(\mu)d\mu = \int_{-1}^{1} \hat{u}\cos^2(\varphi)d\varphi \tag{3.3}$$

(3.3) shows that the first Legendre coefficient can be obtained from a knowledge of the zonal wind, and that it is proportional to the total angular momentum. Since we are trying to compute the zonal wind among other quantities, the formula (3.3) is of no direct use in our case. On the other hand we may argue that the steady state that we are seeking is the result of an atmospheric development starting from a state of rest where the angular momentum is zero. Since the total angular momentum is nearly conserved it should not deviate too much from zero in the steady state. We shall therefore set the first Legendre component of the vorticity equal to zero with the implication that the solution that we shall obtain will have the same property.

The procedure to be used in the remaining parts of the calculations will be to specify the heating decomposed in series of Legendre polynomials. The potential vorticity equations will then be used to obtain the Legendre coefficients of the geopotential. For this purpose we employ the quasi-geostrophic equations where the vorticity is expressed in the geopotential. For reasons of consistency it is necessary to use the vorticity in the form:

$$\varsigma_g = \frac{1}{f_o}\nabla^2\phi \tag{3.4}$$

where f_o is a constant value of the Coriolis parameter ensuring that the area integral of the vorticity vanishes. The steady state potential vorticity equations for levels 1 and 3 are then:

$$\nabla \cdot [\nabla^2\phi_1 - q^2\phi_T)\vec{v}_1] = -\epsilon_T\nabla^2\phi_T - C_hH$$

$$\nabla \cdot [(\nabla^2\phi_3 + q^2\phi_T)\vec{v}_3] = \epsilon_T\nabla^2\phi_T - \epsilon_4\nabla^2\phi_4 - C_hH \tag{3.5}$$

where we have used the standard way of incorporating the friction, and where

$$C_h = \frac{R}{C_p}\frac{-\Omega^2}{\sigma P^2} \tag{3.6}$$

In the equations (3.5) we introduce the parameterizations developed in section 2 and thereafter the Legendre series for the dependent variables are used to obtain the equations for the Legendre coefficients for the Laplacians of the two geopotential fields. Denoting these Legendre coefficients by $Z_1(n)$ and $Z_2(n)$ we obtain after some calculations two linear equations that may be written in the form:

$$a(n)Z_1(n) - bZ_3(n) = -h(n)$$

$$-cZ_1(n) + d(n)Z_3(n) = h(n) \tag{3.7}$$

where the right hand sides contain the Legendre coefficients of the heating, while the coefficients in the equations are defined by

$$a(n) = [n(n+1) + 1/2\lambda^2]\hat{K}_1 + 1/2\hat{\epsilon}_T$$

$$b = 1/2\hat{\epsilon}_T + 1/2\lambda^2\hat{K}_1$$

$$c = 1/2\lambda^2\hat{K}_3 + 1/2\hat{\epsilon}_T + 1/2\hat{\epsilon}_4$$

$$d(n) = [n(n+1) + 1/2\lambda^2]\hat{K}_3 + 1/2\hat{\epsilon}_T + 3\hat{\epsilon}_4/2 \tag{3.8}$$

The equations (3.7) are solved for each n. The recursion relations will give the coefficients for vorticity satisfying the simplified balance equation. We may then proceed to calculate some additional quantities.

The zonal winds may be obtained at levels 1 and 3 from the vorticity by noting that the zonally averaged vorticity is related to the zonally averaged u-component by the relation:

$$\varsigma_z = -\frac{1}{a\cos(\varphi)}\frac{\partial u_z \cos(\varphi)}{\partial \varphi} \tag{3.9}$$

The resulting zonal winds may be obtained either by a direct integration of (3.9) or by using the same equation to calculate the Legendre coefficients of the zonally averaged winds. The same option is available for the other parameters. The zonally averaged vertical velocity is computed from the steady state form of the thermodynamic equation using the known geopotential and the heating. Using the zonally averaged continuity equation it is then possible to compute the zonally averaged meridional wind component.

Finally we mention that certain aspects of the energetics of the model may be computed. It will be seen that the generation of zonal available potential energy $G(A_z)$, the conversion from zonal available potential energy to zonal kinetic energy $C(A_z, K_z)$ and the conversion between the zonal and the eddy available potential energies may be computed. In addition, we may compute the conversion from eddy to zonal kinetic energy $C(K_E, K_z)$ and the dissipation of zonal kinetic energy $D(K_z)$. On the other hand, it is not possible to calculate any additional energy quantities related to the eddy energies.

4. Some examples

The theory developed in sections 2 and 3 will in this section be used on some examples. As a first example we shall use a specified heating that is an approximation to the diagnostic calculation carried out by Lawniczak (1970). Figure 1 shows the applied heating in a normalized form in which the value at the equator is set equal to unity. The heating distribution has been assumed to be symmetrical around the equator. It will be seen that the heating is positive in the lower latitudes and negative elsewhere. The heating has two maxima: one at the equator due to

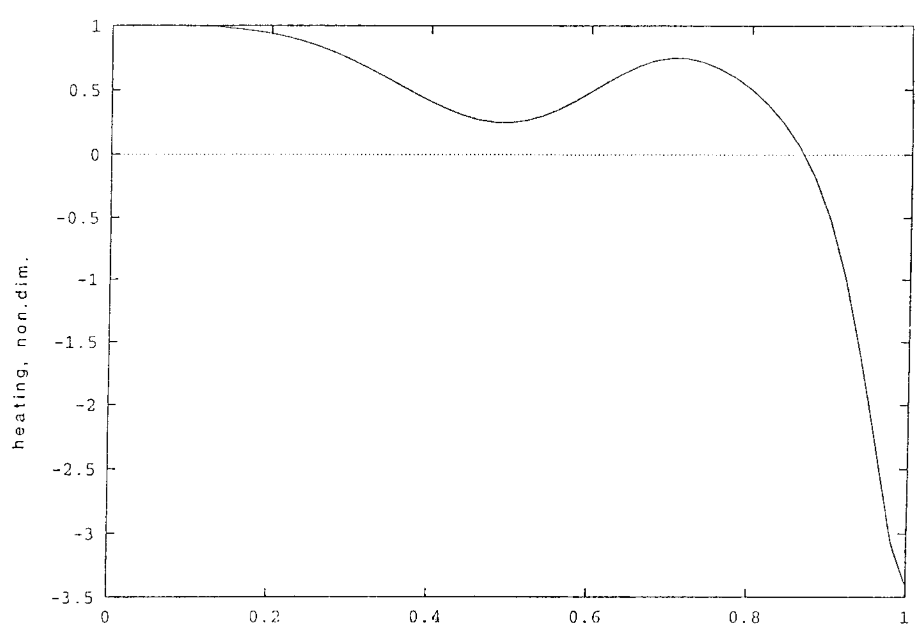

Fig. 1. Heating, normalized to unity at the equator, as a function of the sine of latitude. The curve is an approximation to the heating calculated by Lawniczak (1970)

radiation and another at about 50 degrees north supposedly due to the cyclonic activity in this region. The original curve has been modified slightly to ensure a vanishing area average.

The following results were obtained with parameters as follows:

$$H(0) = 2 \times 10^{-3} J kg^{-1} s^{-1}$$

$$K_1 = 9 \times 10^5 m^2 s^{-1}$$

$$K_3 = 2 \times 10^6 m^2 s^{-1}$$

$$K_H = 1.4 \times 10^6 m^2 s^{-1}$$

$$\epsilon_T = 6 \times 10^{-7} s^{-1}$$

$$\epsilon_4 = 4 \times 10^{-6} s^{-1} \tag{4.1}$$

It is also necessary to specify the coefficient λ^2. This is done by selecting the average value of the Coriolis parameter over the hemisphere (Ω) and otherwise use standard values. We obtain:

$$\lambda^2 = 34.46 \tag{4.2}$$

All calculations with schematic heating distributions have been done using a maximum number of Legendre polynomials equal to 10.

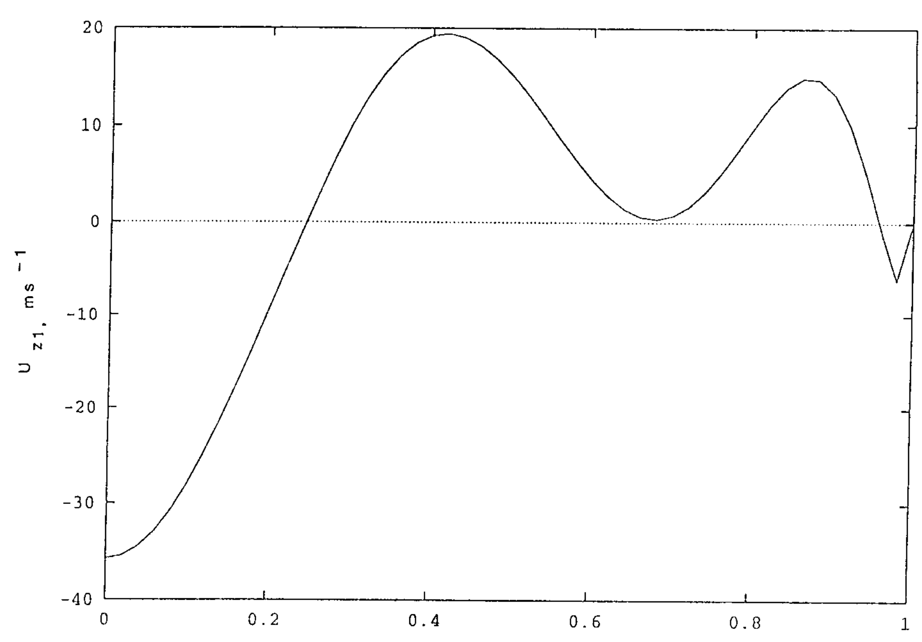

Fig. 2. The zonal wind at the upper level in m s^{-1} as a function of sine of latitude. The heating in Figure 1 has been applied.

Figure 2 gives the zonal wind distribution at the upper level (250 hPa) as a function of μ. Two wind maxima are observed in the westerlies with a somewhat larger wind maximum in the subtropical jet than in the polar jet. Equatorial and polar easterlies are also observed although the easterlies in the low latitudes are stronger than observed. The wind distribution at the lower level (750 hPa) is displayed in Figure 3. It has the same shape as the wind distribution at the upper level, but the speeds are considerably weaker as they should be.

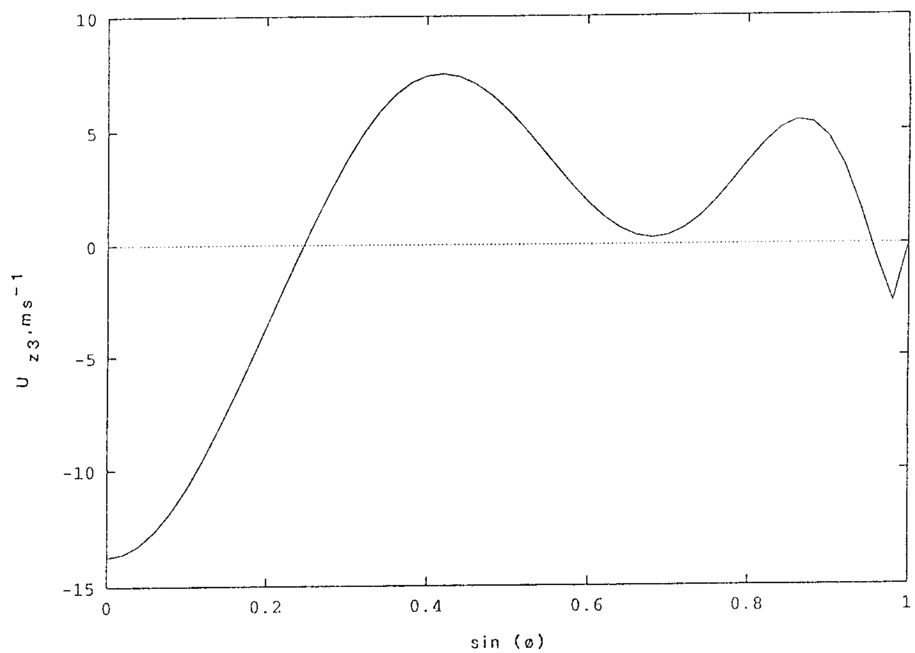

Fig. 3. The zonal wind at the lower level. Same units as in Figure 2.

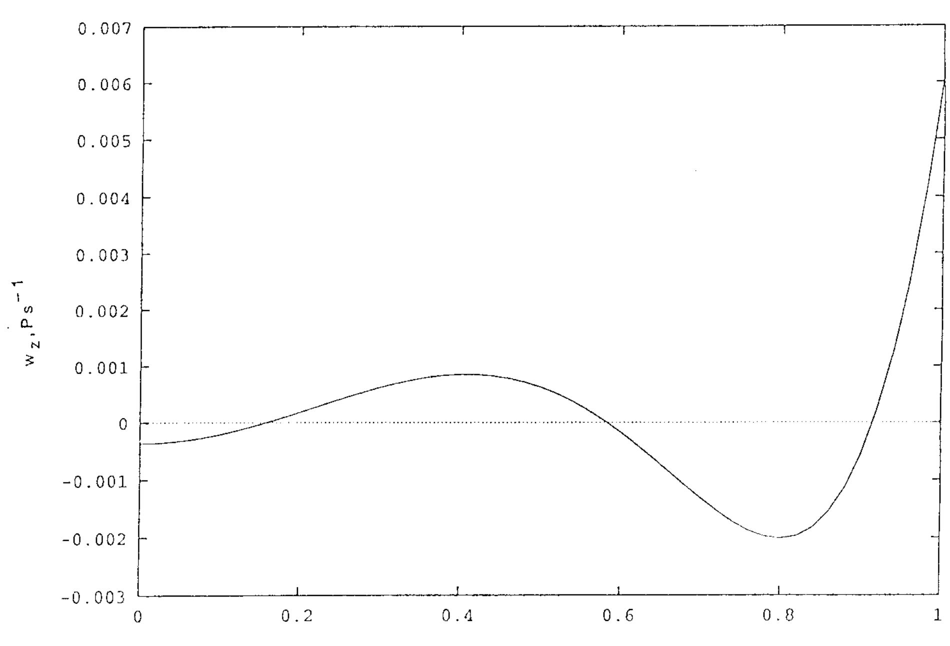

Fig. 4. The zonally averaged vertical p-velocity in P s^{-1} at the middle level as a function of sine of latitude.

The vertical velocity in the p-system (ω) is shown in Figure 4. We notice a weak Hadley cell in the low latitudes, a Ferrel cell in the middle latitude and a rather strong polar Hadley cell. The strength of the polar cell is due to the strong decrease in the cooling over the high latitudes as can be seen from Figure 1. The vertical velocities have a small order of magnitude. In the diagram on Figure 4 they are given in the unit Ps^{-1}. The largest value at the Pole corresponds to no more than about 1 mms^{-1}. The corresponding zonally averaged meridional velocity at the upper level is found in Figure 5, where the unit is ms^{-1}. The corresponding quantity at the lower level is due to the simplicity of the model of the same magnitude, but of the opposite sign. The maximum value is slightly larger than 3 cms^{-1}. While the distribution of the vertical velocities and the zonally averaged meridional velocities are in qualitative agreement with observational studies, the magnitudes are considerably smaller than observed.

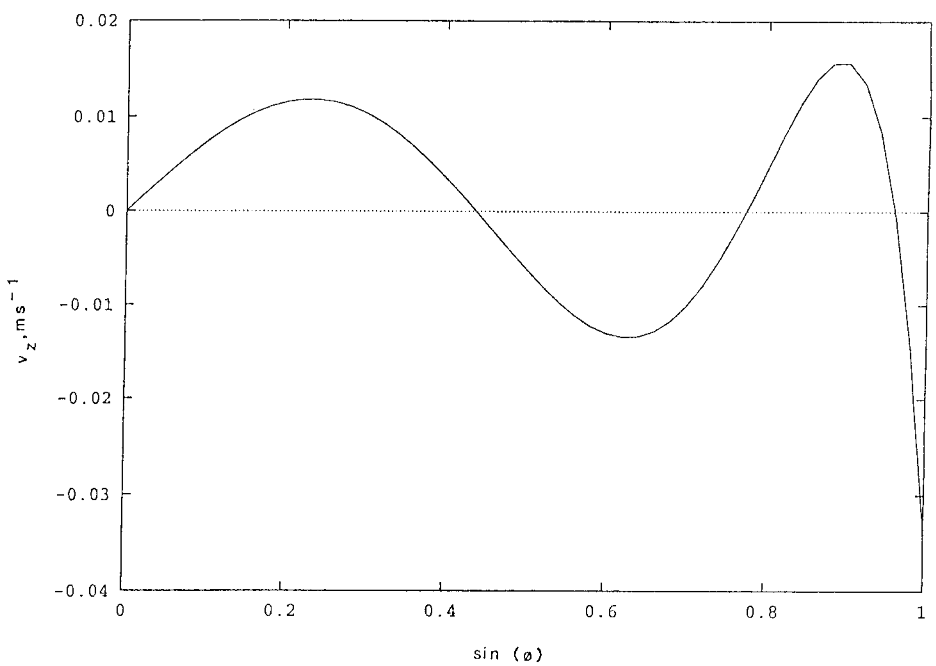

Fig. 5. The mean meridional velocity in m s^{-1} at the upper level as a function of sine of latitude.

Figure 6 and Figure 7 show the meridional transport of vorticity at the upper and lower levels, respectively. As pointed out previously we may consider this transport as the convergence of the momentum transport. A vanishing vorticity transport corresponds therefore to an extremum in the momentum transport. It is evident that the vorticity transports at the two levels have the same shape, but the transport at the upper level is somewhat larger than at the lower level. Figure 8 displays the momentum transports at the two levels with a numerically larger transport at the upper level. A comparison between Figures 6, 7 and 8 and Figures 2 and 3 shows that the convergence of the momentum transport maintains the wind maxima against frictional dissipation.

From the above example based on one of the calculations of the diabatic heating from observations we may thus conclude that the model using the parameterization of the transports of potential vorticity and sensible heat is capable of accounting for some major aspects of the general circulation albeit in a schematical sense only. In judging the results it should be remembered that the heating calculation is based on a single winter months (January, 1969) where the polar as well as the subtropical jets are well developed. It should also be kept in mind that the model assumes constant diffusion coefficients.

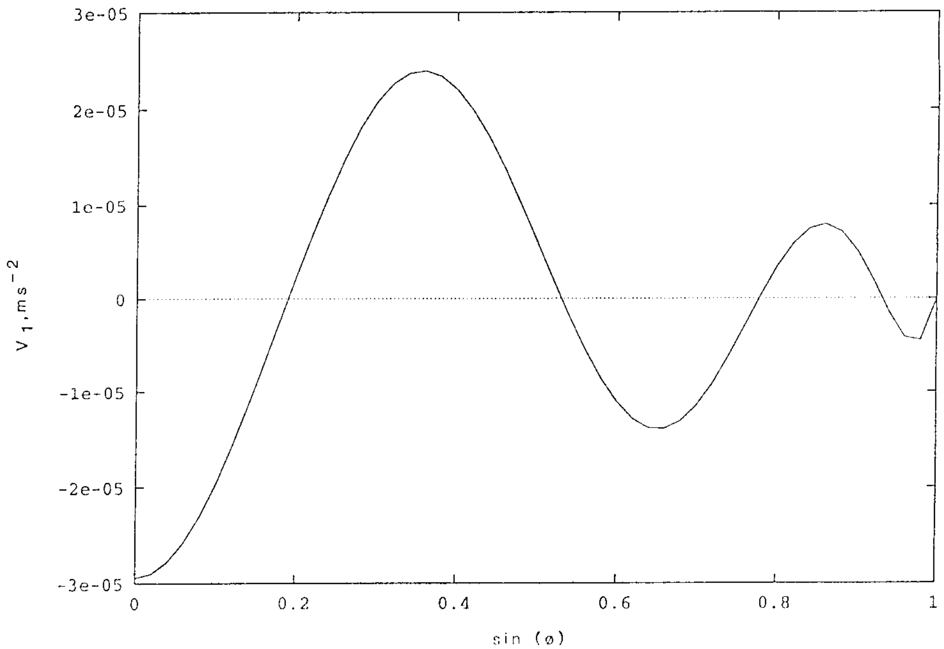

Fig. 6. The vorticity transport at the upper level in the unit: m s^{-2} as a function of sine of latitude.

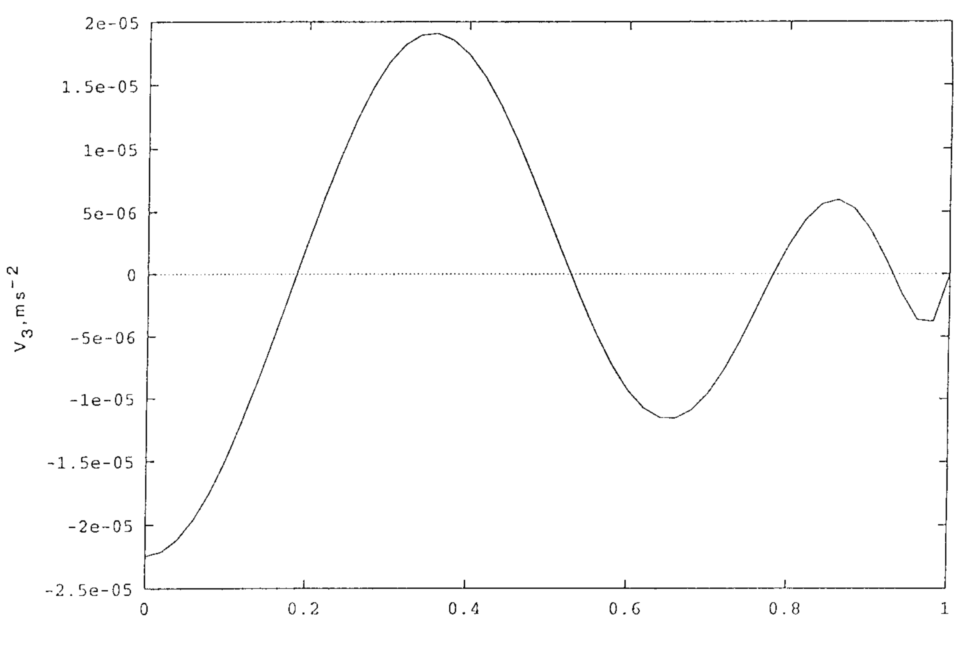

Fig. 7. The vorticity transport at the lower level. Unit and arrangement as in Figure 6.

One may naturally ask if essentially the same results can be obtained with a much simpler specification of the zonally averaged heating. We shall show by an example that this is not the case. The most primitive description of the dependence of the heating on latitude is that heating occurs in the low latitudes and cooling in the remaining part to the North Pole. Such a heating field can be described in several ways, but one example is given by the expression:

$$H(\mu) = 1/2(2 - 9\mu^2 + 7\mu^4) \tag{4.3}$$

Equation (4.3) is normalized in such a way that the value is unity at the equator. The heating (4.3) is shown in Figure 9. To make the point it is sufficient to show the meridional change of the vertical p-velocity. It is shown in Figure 10, while Figure 11 shows the corresponding mean meridional velocity at the upper level. These two figures show a two cell meridional circulation with a direct Hadley cell in the low latitudes and a Ferrel cell in the high latitudes. The region close to the North Pole is therefore characterized by rising motion.

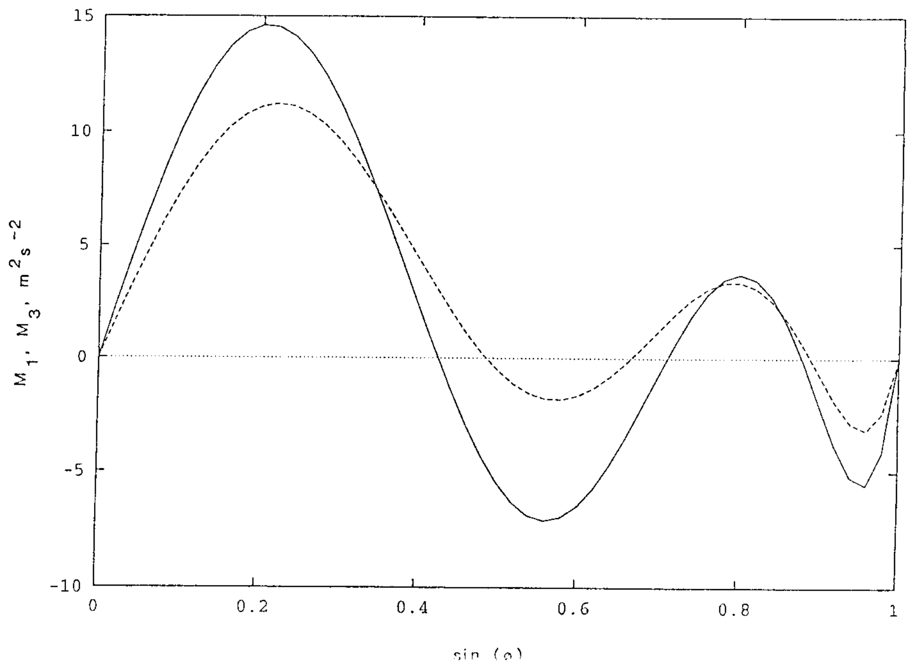

Fig. 8. The momentum transport at the upper level (full curve) and the lower level (dashed curve) in m^2 s^{-2} as a function of latitude.

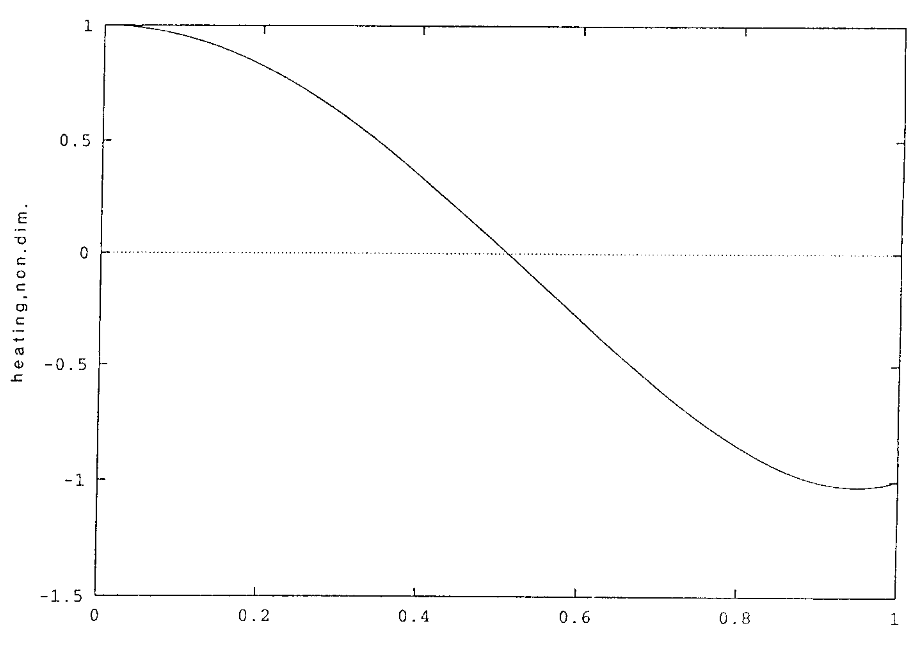

Fig. 9. A simple heating distribution normalized to unity at the equator as a function of sine of latitude.

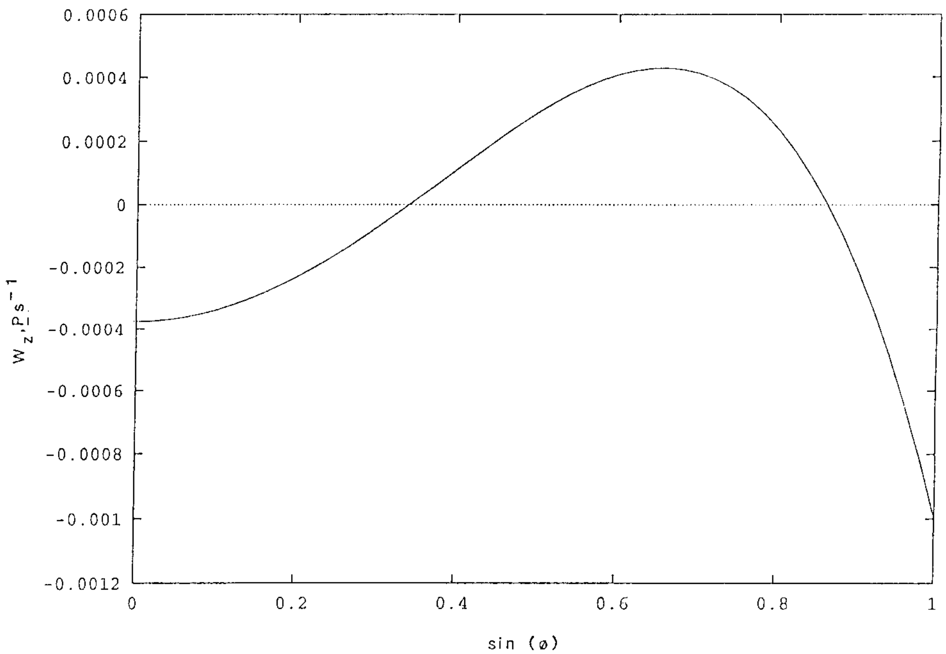

Fig.10. The zonally averaged vertical p-velocity in P s^{-1} at the middle level as a result of the heating in Figure 9. Unit: P s^{-1}

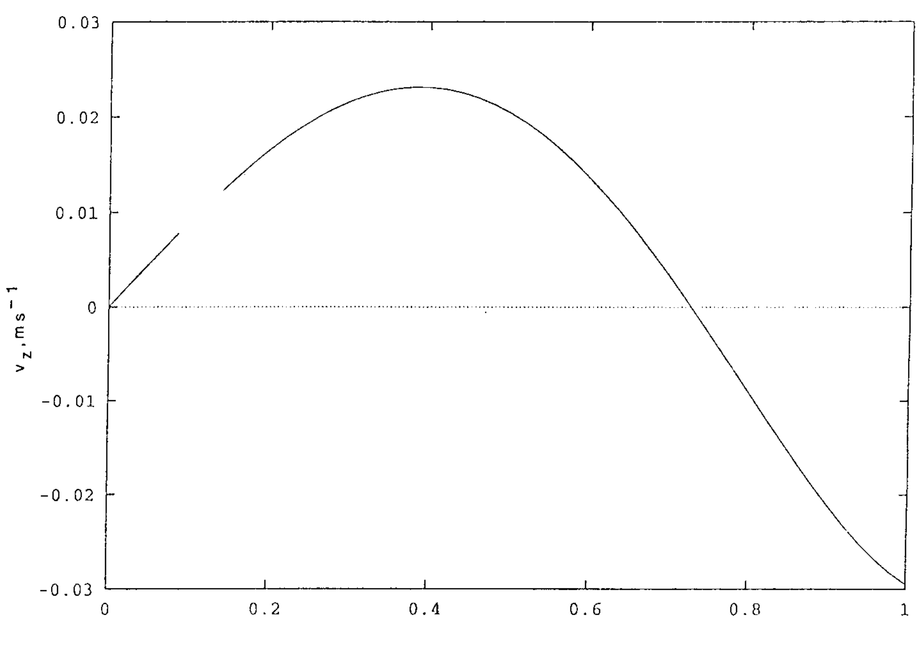

Fig.11. The mean meridional velocity in m s^{-1} at the upper level as a function of sine of latitude as a result of the heating in Figure 9. Unit: m s^{-1}

We list next the computed energy quantities for the main example. The quantities have the following numerical values:

$$
\begin{aligned}
A_z &= 3212 kJm^{-2} \\
K_z &= 2278 kJm^{-2} \\
G(A_z) &= 3.78 Wm^{-2} \\
C(A_z,\ A_E) &= 2.94 Wm^{-2} \\
C(A_z,\ K_z) &= 0.84 Wm^{-2} \\
C(K_E,\ K_z) &= 0.97 Wm^{-2} \\
D(K_z) &= 1.81 Wm^{-2}
\end{aligned}
\tag{4.4}
$$

These values were obtained with the following values of the heating and the diffusion coefficients:

$$
\begin{aligned}
H_o &= 2 \times 10^{-3} Jkg^{-1}s^{-1} \\
K_1 &= 9 \times 10^{5} m^2 s^{-1} \\
K_3 &= 2 \times 10^{6} m^2 s^{-1} \\
K_H &= 1.4 \times 10^{6} m^2 s^{-1}
\end{aligned}
\tag{4.5}
$$

The values given in (4.4) are in good qualitative agreement with observational studies both with respect to sign and to order of magnitude. The quantity most sensitive to the values of the parameters given in (4.5) is the energy conversion from eddy to zonal kinetic energy, because the sign of the vorticity transport and therefore the conversion of the momentum transport is sensitive to these coefficients. If, for example, the diffusion coefficient for potential vorticity at the lower level is decreased from the value given in (4.5) to values closer to the diffusion coefficient for heat, we will obtain a decrease in the conversion and eventually a change of sign.

We may see this from the following listing of values:

K_3, $10^6 m^2 s^{-1}$	$C(K_E, K_z)$, Wm^{-2}
1.9	0.75
1.8	0.54
1.7	0.34
1.6	0.15
1.5	-0.03
1.4	-0.20

The energy conversion listed above is the most sensitive. The other generations, conversions and dissipations maintain the direction for wide variations of the diffusion parameters provided the heating is reasonable. This is understandable from the formulations simply because the system is driven by the heating, making the generation of the zonal available energy positive, just as the dissipation of zonal kinetic energy is positive due to the formulation of the frictional processes in the model. Finally, due to the diffusion of heat the conversion from zonal to eddy available potential energy will also be positive in realistic cases.

5. An improved procedure

The examples in the previous section shows quite clearly that the winds in the low latitudes are of too large a magnitude although the direction is correct. It is possible that this may be due to the assumption that the first Legendre coefficient of the vorticity is set to zero in the previous section. It is easily seen that first Legendre component in the vorticity at a certain level will correspond to a wind distribution with an extremum at the equator and zeroes at the North

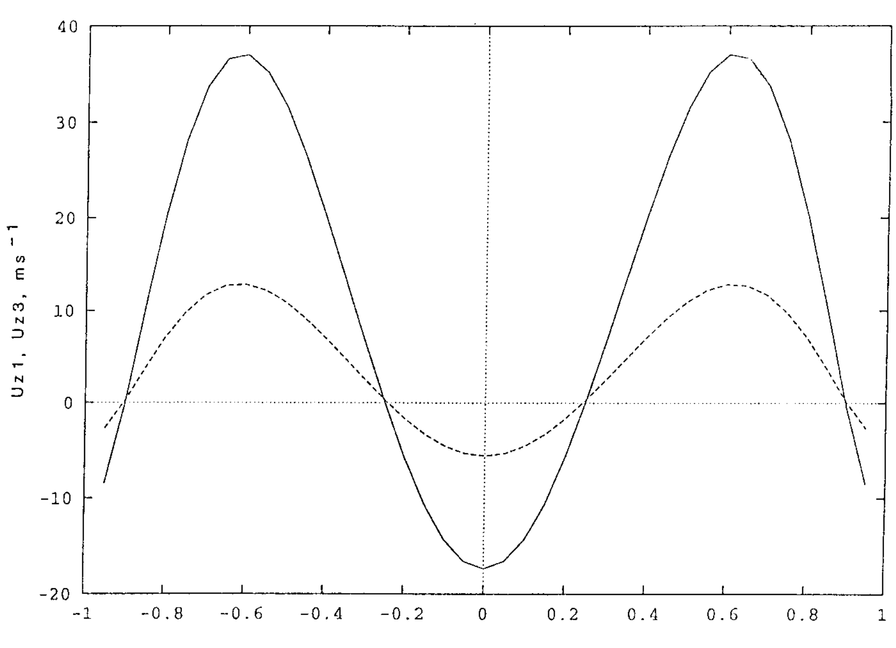

Fig.12. The zonal winds at the upper level (full line) and the lower level (dashed line) for the heating case described by (4.3) as a function of sine of latitude. Unit: m s^{-1}

and the South Poles. Such a component with a westerly wind at the equator would improve the results. Lacking any other methods it was decided to obtain the first coefficient by a slightly inconsistent procedure. This was done by using the standard two level quasi-nondivergent model, but to let the Coriolis parameter vary in such a way that the parity relations were maintained. Since the averaged heating over the globe, i.e. the coefficient corresponding to the Legendre polynomial P_0, vanishes we will then get a relation in which the first coefficient of the vorticity is related to the second coefficient of the heating. This procedure is used only for the first coefficient of the vorticity. The remaining Legendre coefficients are as before computed from the linear balance equation (see Wiin-Nielsen, 1988).

We start by doing the calculation with the new estimate for the first Legendre coefficient for the heating, specified in (4.3), having heating in the low latitudes and cooling in the high latitudes. The constants are those given in (4.1). Figure 12 shows the two windprofiles at the

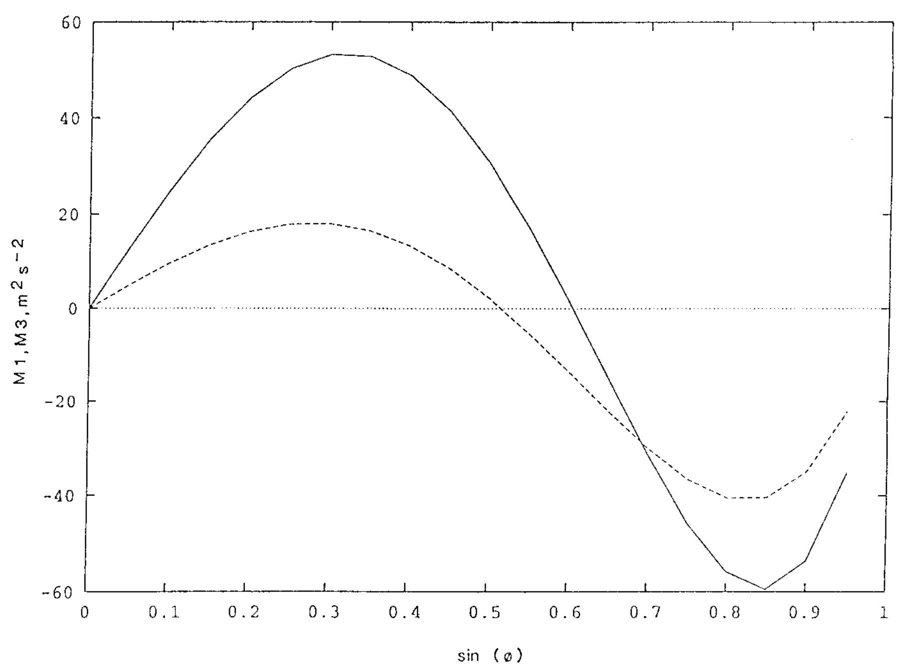

Fig.13. The momentum transports at the upper level (full line) and the lower level (dashed line) as a function of sine of latitude for the same case as Figure 12. Unit: $m^2\ s^{-2}$

upper and lower levels. The number of Legendre functions is in this section reduced to N=6 to concentrate on the large scale features. We notice a single maximum in each hemisphere, but also reduced easterlies in the region around equator. Figure 13 contains the momentum transports at the two levels. There is agreement between the windprofiles and the momentum transports in the sense that the wind maxima are situated at the place of maximum convergence. Figure 14 and Figure 15 show the vertical p-velocity and the mean meridional wind, respectively. As in the previous section we find a single meridional cell of the Hadley type. The energy amounts and the generations and conversions are as follows:

$$Az = 5498kJm^{-2}$$

$$Kz = 2207kJm^{-2}$$

$$G(Az) = 2.94Wm^{-2}$$

$$C(Az,\ Ae) = 2.32Wm^{-2}$$

$$C(Az,\ Kz) = 0.62Wm^{-2}$$

$$C(Ke,\ Kz) = 1.27Wm^{-2}$$

$$D(KT) = 1.24Wm^{-2}$$

$$D(K4) = 0.65Wm^{-2}$$

The amount of zonal available potential energy is somewhat higher than obtained from data studies, but the other figures are reasonable in both direction and magnitude.

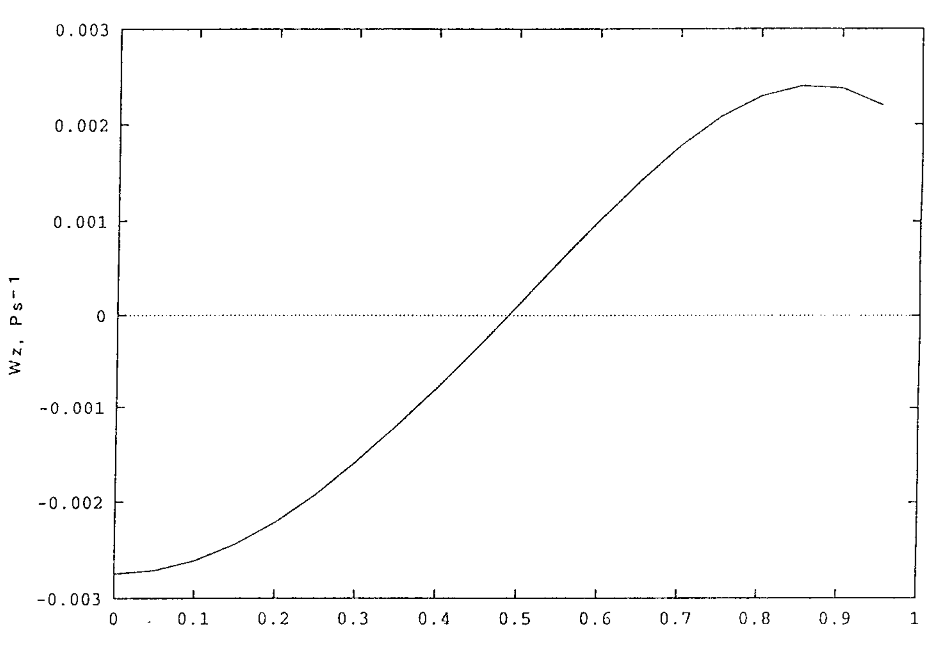

Fig.14. The vertical p-velocity as a function of sine of latitude for the same case as Figure 12. Unit: P s^{-1}

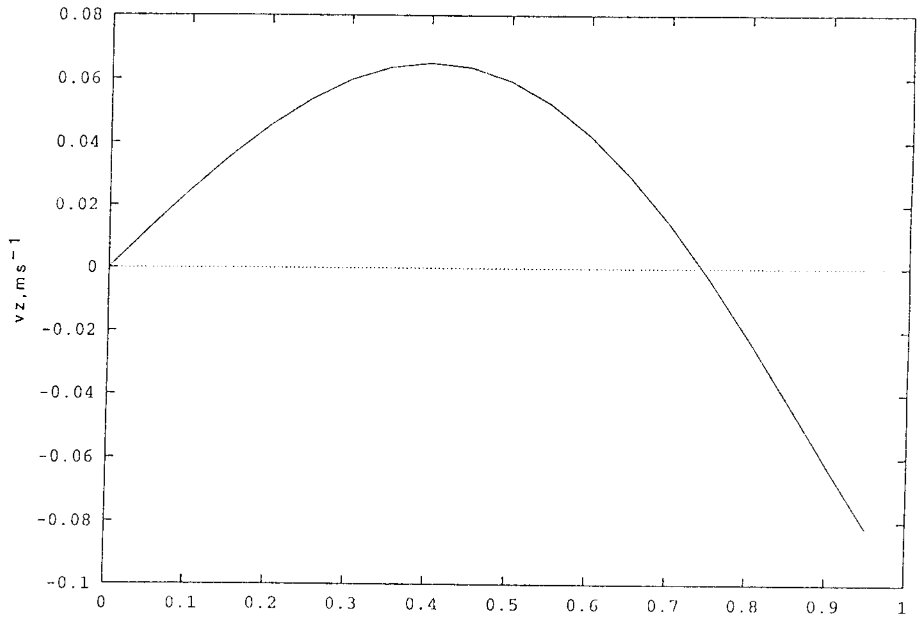

Fig.15. The mean meridional velocity as a function of sine of latitude for the same case as Figure 12. Unit: m s^{-1}

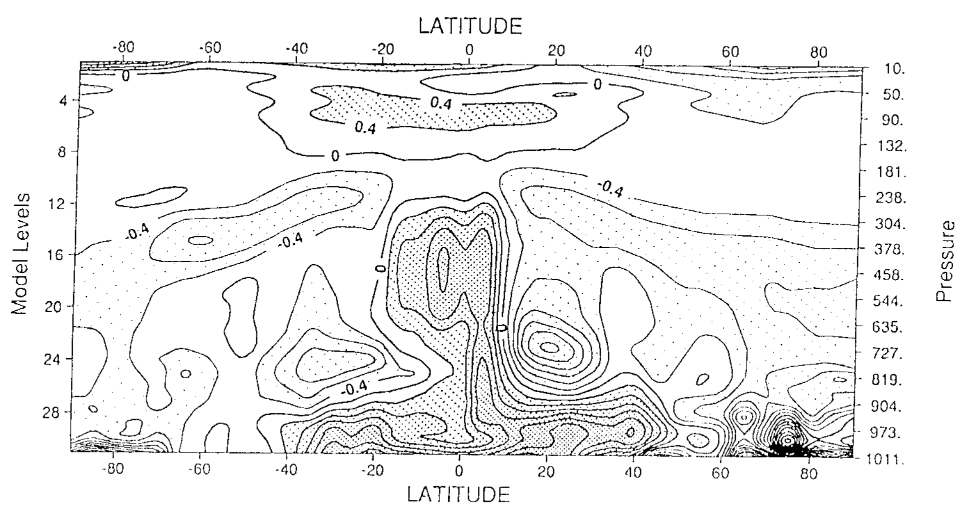

Fig.16. The zonal mean diabatic heating as computed from the ECMWF model as a function of pressure and latitude. The unit is degrees per day. Mean values for January, 1993.

As the next example we have selected the averaged heating for January, 1993 as provided from the European Centre for Medium-Range Forecasts and based on the computation of the diabatic heating from analyses done with an interval of 6 hour. The heating field, displayed in Figure 16, is actually provided at all the pressure levels of the analysis. One will notice from the figure that the heating is mainly a low level feature apart from the tropics, where the convection provide heating all the way into the stratosphere. In our simple two level model the heating is used only at the middle level (500 hPa). The distribution of the vertically averaged heating with respect to latitude is displayed in Figure 17. The maximum heating is found at the equator, when the data have been smoothed to remove two-grid-increment waves, and when the heating has been corrected in such a way that the area averaged heating vanishes. The result is shown in Figure 17 indicating clearly the northern wintertime strong cooling in the high latitudes as compared to the much smaller cooling close to the South Pole. One notices also the decreased cooling in the Southern Hemisphere close to 50 degrees south, while the same feature is found in the Northern Hemisphere at 35 degrees north.

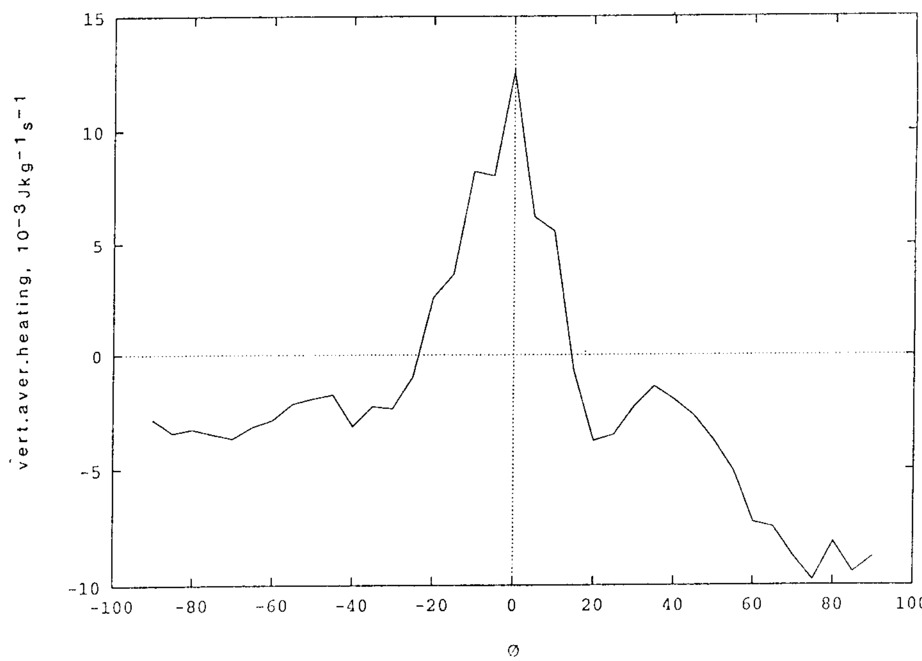

Fig.17. The vertically averaged mean diabatic heating as a function of latitude for January 1993. The unit is 10-3 J kg^{-1} s^{-1}.

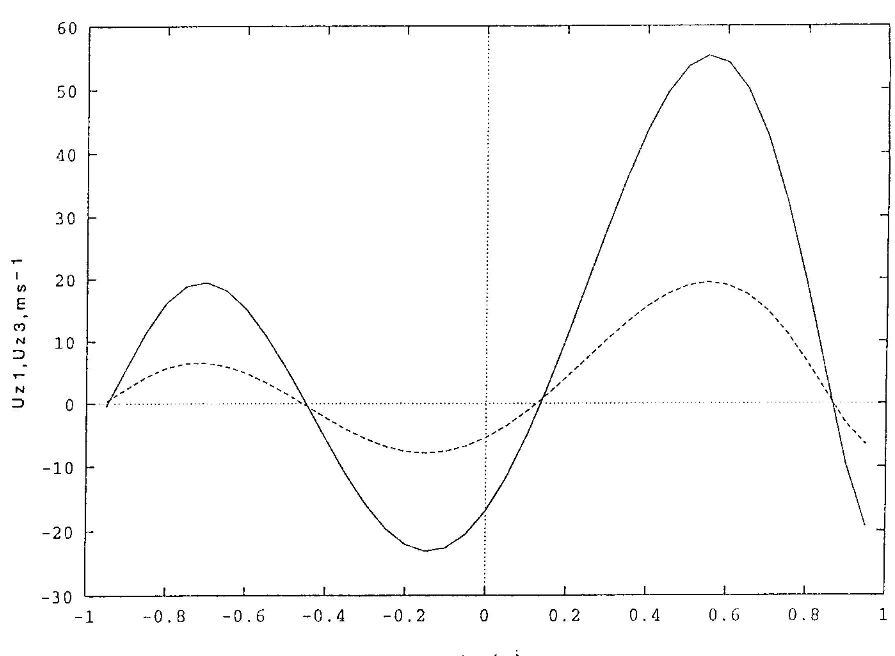

Fig.18. The zonal mean winds at the upper level (full curve) and the lower level(dashed curve) as a function of sine of latitude for the heating described in Figure 17. Unit: m s^{-1}

The wind profiles at the two levels are shown in Figure 18. The strength of the jet stream in the northern latitude is more than 50 m per sec as compared to about 20 m per sec in the Southern Hemisphere. The jet in the winter hemisphere is somewhat closer to the equator than the jet in the summertime hemisphere. The equatorial easterlies are wider in the summer hemisphere, where they cover the latitude belt from the equator to almost 30 degrees south. Weak polar easterlies are found in both hemispheres.

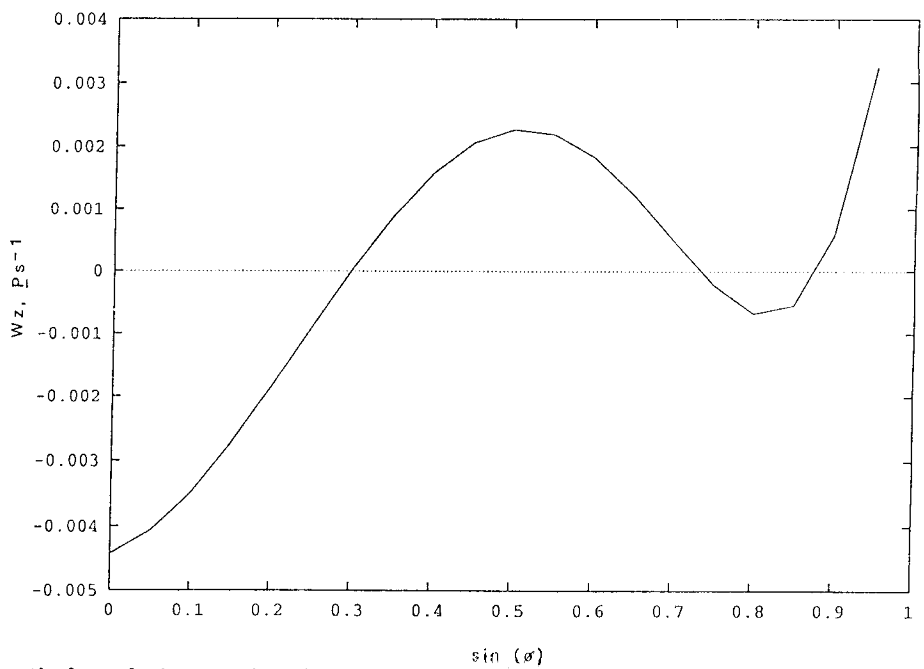

Fig.19. The vertical p-velocity as a function of sine of latitude for the heating given in Figure 17. Unit: P s^{-1}.

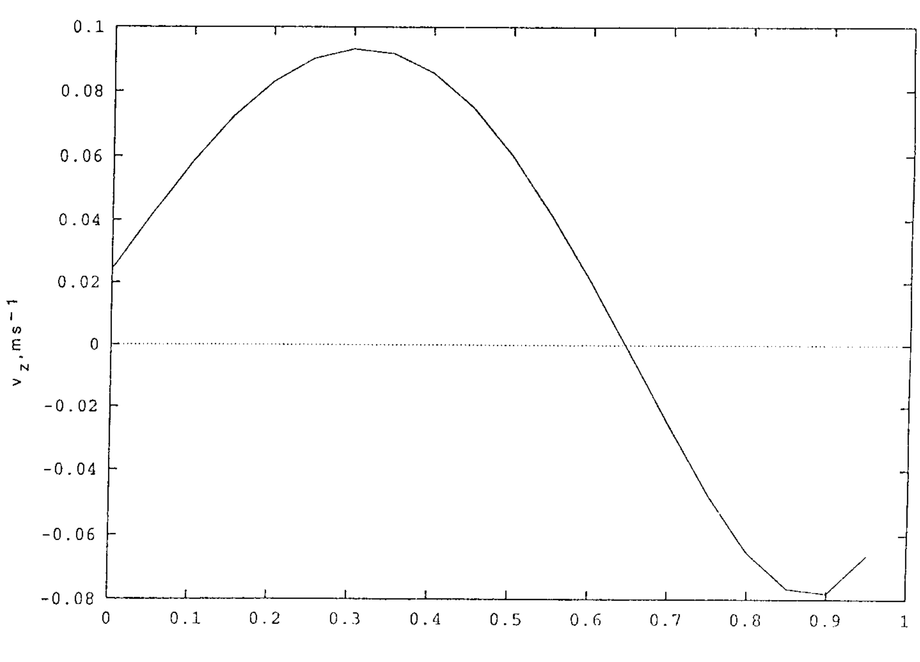

Fig. 20. The mean meridional velocity as a function of sine of latitude for the heating described in Figure 17. Unit: m s^{-1}.

The zonally averaged vertical p-velocity is displayed in Figure 19 for the Northern Hemisphere. It shows rising motion from the equator to about 17.5 degrees north followed by sinking motion to about 45 degrees north. Rising motion is found again from that latitude to about 60 degrees north, where it is replaced by sinking motion to the North Pole. The mean meridional circulation corresponding to Figure 19 is shown in Figure 20. The reason that this curve does not clearly indicate the three cell pattern as seen in Figure 18 is due to the fact that a zero in the vertical velocity corresponds to an extremum in the curve for the mean meridional velocity. With the weak indirect cell in the middle latitudes the mean meridional velocity does not change sign.

The energy quantities for this case are:

$$Az = 6712 kJm^{-2}$$

$$Kz = 2727 kJm^{-2}$$

$$G(Az) = 3.61 Wm^{-2}$$

$$C(Az,\ Ae) = 2.97 Wm^{-2}$$

$$C(Az,\ Kz) = 0.64 Wm^{-2}$$

$$C(Ke,\ Kz) = 4.47 Wm^{-2}$$

$$D(Kt) = 2.38 Wm^{-2}$$

$$D(K4) = 2.73 Wm^{-2}$$

We notice in this case again the rather large value of the zonal available potential energy. The conversion from eddy to zonal kinetic energy is also somewhat larger than values obtained from observational studies. It should be remembered that the example is based on heating data from a specific month.

The transports of vorticity and momentum were also investigated, but they reveal nothing new as compared to the previous examples.

6. Approach to the steady states

It is of interest to estimate how long time it takes to approach the steady state, if the system is in an arbitrary state. Since the equations are linear in the present case, such a measure can be obtained by analytical means. To treat the problem it is first of all necessary to add time dependence to the steady state equations treated so far.

We shall as before work with the quasi-geostrophic equations. Denoting the Laplacian for the zonally averaged quantities by L, i.e.

$$L = \frac{\partial}{\partial \mu}[1 - \mu^2]\frac{\partial}{\partial \mu} \tag{6.1}$$

and working with a nondimensional time

$$\tau = t\Omega \tag{6.2}$$

we may write the quasi-geostrophic equations for the two level model in the form:

$$\frac{\partial}{\partial \tau}[L(\hat{\Phi}_1) - \lambda^2 \hat{\Phi}_T] - \hat{K}_1 L[L(\hat{\Phi}_1) - \lambda^2 \hat{\Phi}_T] = \hat{\epsilon}_T L(\hat{\Phi}_T) - \hat{H} \tag{6.3}$$

$$\frac{\partial}{\partial \tau}[L(\hat{\Phi}_3) + \lambda^2 \hat{\Phi}_T] - \hat{K}_3 L[L(\hat{\Phi}_3) + \lambda^2 \hat{\Phi}_T] = \hat{\epsilon}_T L(\hat{\Phi}_T) - \hat{\epsilon}_4 L(\Phi_4) + \hat{H} \tag{6.4}$$

where we have used the same notations and the same scaling as in the previous sections. The equations (6.3) and (6.4) may then be written in wave number space using the identical expansions as before. Thereafter we subtract the steady state equations from both equations and obtain equations for the deviations from the steady states. The result is:

$$\alpha(n)\frac{dx_1(n)}{d\tau} - \beta(n)\frac{dx_3(n)}{d\tau} = -a(n)x_1(n) + bx_3(n)$$

$$-\beta(n)\frac{dx_1(n)}{d\tau} + \alpha(n)\frac{dx_3(n)}{d\tau} = cx_1(n) - d(n)x_3(n) \tag{6.5}$$

where the Legendre coefficients are denoted by $x(n).a(n)$, b, c and $d(n)$ have been defined before. The new notations are:

$$\alpha(n) = 1 + \beta(n)$$

$$\beta(n) = 1/2\frac{\lambda^2}{n(n+1)} \tag{6.6}$$

A solution of the equations above may be obtained by assuming that they are of the form:

$$x_1(n) = \hat{x}_1(n)e^{\nu_n t}$$

$$x_3(n) = \hat{x}_3(n)e^{\nu_n t} \tag{6.7}$$

We obtain then two homogeneous linear equations with $\nu(n)$ as the unknown. Setting the determinant equal to zero we get a second degree equation from which we obtain two solutions for the frequency. They will both be negative indicating as expected that the computed steady state for the zonal structure is stable. When an e-damping time is computed from the numerically larger frequency we get a measure of how fast the steady state is approached. The e-damping time is a function of n, and it is seen from Figure 12 that the largest scales take the longest time to come close to the steady state. This means that we probably never see a steady state in reality, because the forcing is changing in time preventing a real approach to the steady state.

7. Concluding remarks

The simplified theory of certain aspects of the general circulation of the atmosphere presented in this and earlier papers is far from complete simply because it cannot describe the evolution of the eddies and their intensity. It is only capable of accounting for some major aspects of the zonal structure of the atmosphere due to the parameterizations of the various meridional eddy transport processes. In this sense the theory may be seen as a continuation of the classical efforts to describe the major physical processes maintaining the zonal structure of the winds and temperature fields. The difference between the present and classical models is the major emphasis on incorporating the effects of the eddies on the zonal structure, while the attempts made by Hadley, Ferrel and others tried to explain the zonal structure considering the symmetrical structure only. As any empirical theory it relies on the heuristic assumption that only quasi-conservative quantities may be described as large-scale diffusion processes. In the theory this principle is applied to quasi-geostrophic potential vorticity and to potential temperature, while the momentum transport by the eddies is obtained in an indirect manner. In formulating the parameterizations it is also important to ensure that certain integral constraints valid for the zonal structure in general are satisfied for the parameterization as well.

The theory is capable of accounting for the general structure of the tropospheric zonal wind systems and for the inherent mean meridional circulation provided the assumed heating is realistic. The derived transports of heat and momentum are also in good agreement with observational studies.

In the present version of the theory we have emphasized simplicity by using a two- level quasi-geostrophic model and constant values of the diffusion coefficients. In specifying the values of the coefficients we have used the results of calculations based on observations, but we have also to a limited extent investigated the sensitivity of the results to the numerical values of the coefficients. An expansion of the theory to multi-level quasi-geostrophic models is possible, but requires a specification of the values of the diffusion coefficients at additional levels. It should also be possible to expand the theory to quasi-geostrophic models with a continuous variation of the dependent variables in the vertical direction. In the latter case it will probably be advantageous to use vertical structure functions.

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