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Reaction of two simple nonlinear dynamical systems to constant forcing

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RESUMEN

En general, los sistemas dinámicos no lineales (NDS's) descritos por sistemas de ecuaciones diferenciales ordinarias no lineales son sensibles a los cambios estructurales de éstas. Aún constantes simples aditivas pueden cambiar el comportamiento global de un sistema dado, por ejemplo (1.4) mostrado a continuación.

Para estudiar directamente este efecto escogimos dos (NDS's) muy conocidos: el sistema clásico de Lorenz (1.2), (1.3) y el sistema de Wiin-Nielsen (1.6) que tienen ambos varias aplicaciones meteorológicas. Más aún, postulamos y damos una respuesta al problema inusitado: ¿Cuál deberá ser el forzamiento aditivo constante de estos sistemas, a fin de tener un número deseado de puntos fijos reales con unas propiedades de estabilidad deseadas? Se incluyen también algunos resultados numéricos preliminares.

ABSTRACT

Generally, the nonlinear dynamical systems (NDSs) described by systems of nonlinear ordinary differential equations, are sensitive to structural changes of the latter. Even simple additive constants can change the global behaviour of a given system, e.g. (1.4) below.

To study directly this effect we choose two popular (NDSs) - the Lorenz classical system (1.2), (1.3) and the Wiin-Nielsen system (1.6), both having various meteorological applications. Moreover, we ask and give an answer to the nonstandard problem - what should be the additive constant forcing of these systems in order to have a desired number of real fixed points with desired stability properties? Some preliminary numerical results are also presented.

1. Introduction

An autonomous quadratically nonlinear dynamical system (NDS) can be written in the most general form as

$$\dot{X}_i = \sum_{j,k} a_{ijk} X_j X_k + \sum_j b_{ij} X_j + c_i \tag{1.1}$$

where $X_\alpha = X_\alpha(t)$, $\dot{X}_\alpha = \mathrm{d}X_\alpha/\mathrm{d}t$, $(a,\ b,\ c)$ are tensor quantities of different rank independent of time (t). Obviously, such systems do not have trivial or zero $(\overline{X}_\alpha = 0)$ stationary $(\dot{X}_\alpha = 0)$ solutions (fixed points in the phase space spanned by X_α). The great majority of the widely known NDSs from purely mathematical or applied problems are of the type (1.1) with $c_i = 0$, i.e. homogeneous systems of ordinary differential equations (ODE). Among them, the classical Lorenz system (1963), hereafter referred to as L-system, occupies a central place.

$$\dot{X} = -\sigma X + \sigma Y$$

$$\dot{Y} = rX - Y - XZ$$

$$\dot{Z} = bZ + XY \tag{1.2}$$

where $\sigma,\ b,\ r$ are positive constant parameters. A special book was published about it (Sparrow, 1982). This system can be found in quite different fields - geophysical fluid dynamics including first of all meteorology, laser physics, astrophysics, etc. (Panchev, 1996). It is considered something like an icon of the nonlinear science.

Obviously (1.2) is a homogeneous system of ODEs. However, in a number of cases (1.2) appears in the form (1.1), i.e.

$$\dot{X} = -\sigma X + \sigma Y + \sigma_o$$

$$\dot{Y} = rX - Y - XZ + r_o$$

$$\dot{Z} = -bZ + XY + b_o \tag{1.3}$$

where $\sigma_o,\ b_o,\ r_o$ are three additional parameters corresponding to c_i in (1.1). As typical examples we refer to the recent publications of Palmer (1993, 1995). To our knowledge, a direct study of the reaction of (1.2) to such kind of forcing has not been done so far. This is one of the goals of the present paper.

Generally, the NDSs are sensitive to structural changes of the equations and it cannot be predicted in advance what will be the global response of the system to such changes, even in the case of additive constants like c_i in (1.1) and particularly in (1.3). To illustrate this statement we refer to another NDS proposed also by Lorenz (1984) as a simplest possible low-order model of the general atmospheric circulation:

$$\dot{X} = -Y^2 - Z^2 + a(F - X)$$

$$\dot{Y} = XY - bXZ - Y + G$$

$$\dot{Z} = bXY + XZ - Z \tag{1.4}$$

where a, b, F, G are constants. For some values of these parameters the system (1.4) exhibits chaotic behaviour in time, whereas if $G = 0$ the latter is in principle impossible. Actually, with $G = 0$ and $Y^2 + Z^2 = U$ one obtains

$$\dot{X} = -U + a(F - X)$$

$$\dot{U} = 2U(X - 1) \tag{1.5}$$

However, a two-dimensional autonomous NDS cannot have a chaotic solution.

A second NDS for comparative study here will be the following one, labelled as the W-N system.

$$\dot{X} = XY + YZ + m_1$$

$$\dot{Y} = 2XZ - X^2 + m_2$$

$$\dot{Z} = -3XY + m_3 \tag{1.6}$$

where m_k are constants. As shown in Wiin-Nielsen (1996) it can be derived from the one-dimensional "advection" equation

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = f(x). \tag{1.7}$$

The latter equation has been used by many authors to study qualitatively the principal role of the quadratic nonlinearity in various fluid dynamic problems (nonlinear waves, turbulence, predictability, numerical schemes, etc.). This fact motivates our choice of the system (1.6) since it is a low-order approximation to (1.7).

For comparison we will need also some basic information about the unforced systems (1.2) and (1.6) at $m_k = 0$. The original Lorenz system (1.2) has a zero fixed point (0, 0, 0), which is stable at $r < 1$ and unstable at $r > 1$, and two nonzero points only at $r > 1$

$$(\overline{X},\ \overline{Y},\ \overline{Z}) = (\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1) \tag{1.8}$$

They are stable at

$$1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \tag{1.9}$$

and become unstable at $r > r_H$, provided that $\sigma > b+1$. As a result, chaotic behaviour is observed.

Traditionally it is assumed $\sigma = 10,\ b = 8/3$. Then $r_H \approx 24.74$.

As to the second system (1.6), it is readily seen that at $m_k = 0$

$$X^2 + Y^2 + Z^2 = R_o^2 = const \tag{1.10}$$

where R_o^2 is determined by the initial conditions ($t = 0$) about $X,\ Y,\ Z$. Because of (1.10) chaotic regime is impossible. The system possesses six (three pairs) fixed points as follows:

$$\begin{aligned}
&(0,\ \pm R_o,\ 0) && -\text{ spiral nodes,}\\
&(\pm 2R_o/\sqrt{5},\ 0,\ \pm R_o/\sqrt{5}) && -\text{ centres,}\\
&(0,\ 0, \pm R_o) && -\text{ saddle points.}
\end{aligned} \tag{1.11}$$

Without loss of generality one can let $R_o = 1$.

In section 2, organized in subsections, we study the stationary and stability properties of the system (1.3). Section 3 is devoted to the W-N system (1.6). In both cases some numerical computations and discussions supplement the derivations.

The modified systems (1.3) and (1.6) we are going to study here are not intended to model some particular meteorological or other phenomena. Rather, they are viewed as mathematical equations only.

2. The forced Lorenz system

2.1. General solutions - Letting in (1.3) $\dot{X} = \dot{Y} = \dot{Z} = 0$ we get

$$\overline{Y} = \overline{X} - \frac{\sigma_o}{\sigma}, \qquad \overline{Z} = \frac{1}{b}(b_o + \overline{X}\ \overline{Y}), \tag{2.1}$$

$$\overline{X}^3 - \frac{\sigma_o}{\sigma}\overline{X}^2 + [b_o - b(r-1)]\overline{X} - b(r_o + \frac{\sigma_o}{\sigma}) = 0 \tag{2.2}$$

The stability properties of the fixed points (2.1), (2.2) are governed by the eigenvalue cubic equation

$$\lambda^3 + K\lambda^2 + L\lambda + M = 0 \tag{2.3}$$

where

$$\begin{aligned}
K &= \sigma + b + 1,\\
L &= \sigma(b+1) + b + \overline{X}^2 - \sigma(r - \overline{Z}),\\
M &= \sigma[b + \overline{X}^2 + \overline{X}\ \overline{Y} - b(r - \overline{Z})].
\end{aligned} \tag{2.4}$$

Let δ_1, δ_2, δ_3 be the roots of the cubic equations (2.2), i.e. $(\overline{X}-\delta_1)(\overline{X}-\delta_2)(\overline{X}-\delta_3)=0$ or

$$\overline{X}^3-(\delta_1+\delta_2+\delta_3)\overline{X}^2+(\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1)\overline{X}-\delta_1\delta_2\delta_3=0 \tag{2.5}$$

Normally $(\delta_1, \delta_2, \delta_3)$ and $(\lambda_1, \lambda_2, \lambda_3)$ have been searched for provided all parameters entering (1.3) are given. Unlike this approach we formulate here the inverse problem: what should σ_o, b_o, r_o be in order to have a desired number of real fixed points δ_k with desired stability properties λ_k?

Equating (2.5) to (2.2) we get an answer to the first question

$$\sigma_o=\sigma(\delta_1+\delta_2+\delta_3),$$

$$b_o=b(r-1)+\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1,$$

$$r_o=\frac{1}{b}\delta_1\delta_2\delta_3-(\delta_1+\delta_2+\delta_3). \tag{2.6}$$

Thus, in the general case the fixed points are

$$P_1\equiv(\overline{X}_1,\overline{Y}_1,\overline{Z}_1)=(\delta_1,-\delta_2-\delta_3,r-1+\frac{1}{b}\delta_2\delta_3)$$

$$P_2\equiv(\overline{X}_2,\overline{Y}_2,\overline{Z}_2)=(\delta_2,-\delta_3-\delta_1,r-1+\frac{1}{b}\delta_3\delta_1)$$

$$P_3\equiv(\overline{X}_3,\overline{Y}_3,\overline{Z}_3)=(\delta_3,-\delta_1-\delta_2,r-1+\frac{1}{b}\delta_1\delta_2) \tag{2.7}$$

and also

$$L=b(\sigma+1)+\frac{\sigma}{b}[b_o-b(r-1)]+(1+\frac{\sigma}{b})\overline{X}^2-\frac{\sigma_o}{b}\overline{X},$$

$$M=\sigma[b_o-b(r-1)]+3\sigma\overline{X}^2-2\sigma_o\overline{X}, \tag{2.8}$$

while $K=\sigma+b+1$ does not depend on the forcing parameters σ_o, b_o, r_o. Traditionally $\sigma=10$, $b=8/3$, so that $K=41/3$ and this value will be used hereafter. In the calculations we shall use another traditional value: $r=28$ for which the solution to (1.2) is chaotic. Therefore, primary bifucation parameters in our approach are $(\delta_1, \delta_2, \delta_3)$ or according to (2.6): (σ_o, b_o, r_o).

The cubic equation (2.2) can have three unequal real roots $\delta_1\neq\delta_2\neq\delta_3$, two real roots δ_1, $\delta_2\equiv\delta_3$ and one real triple root $\delta_1\equiv\delta_2\equiv\delta_3=\delta$ or δ_1, $\delta_{2,3}=\delta_R\pm i\omega$. We are interested in the stability properties of the real fixed points (2.7) only. As it is known, a fixed point $(\overline{X}, \overline{Y}, \overline{Z})$ will be stable if the coefficients in (2.3) satisfy simultaneously the following inequalities

$$K>0, \quad M(\delta_1, \delta_2, \delta_3)>0, \quad KL(\delta_1, \delta_2, \delta_3)>M \tag{2.9}$$

Otherwise the fixed point is unstable. If $KL=M$ a change of stability is observed. In the case $\lambda = \lambda_R + i\lambda_I$ this means $\lambda_R = 0$ (Hopf bifurcation). Since δ_1, δ_2, δ_3 (including δ_R, ω) may be chosen freely, several different cases are possible. Some of the results which we need for the next discussion are summarized in the Table 1.

$\overline{X}_1 = \delta_1$ $\overline{X}_2 = \delta_2$ $\overline{X}_3 = \delta_3$	**A** $\delta_1 \neq \delta_2 \neq \delta_3$ – REAL	**B** $\delta_1, \delta_2 \equiv \delta_3 = \delta_R$ – REAL	**C** δ_1 – REAL $\delta_{2,3} = \delta_R + i\omega$	**D** $\delta_1 \equiv \delta_2 \equiv \delta_3 = \delta$ – REAL
$\sigma_0 =$	$\sigma(\delta_1+\delta_2+\delta_3)$	$\sigma(\delta_1+2\delta_R)$	$\sigma(\delta_1+2\delta_R)$	$3\sigma\delta$
$b_0 =$	$b(r-1)+\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1$	$b(r-1)+\delta_R^2+2\delta_R\delta_1$	$b(r-1)+2\delta_R\delta_1+Q^2$	$b(r-1)+3\delta^2$
$r_0 =$	$\frac{1}{b}\delta_1\delta_2\delta_3-(\delta_1+\delta_2+\delta_3)$	$\frac{1}{b}\delta_1\delta_R^2-(\delta_1+2\delta_R)$	$\frac{\delta_1}{b}Q^2-(\delta_1+2\delta_R)$	$\frac{1}{b}\delta^3-3\delta$
$L_1 =$	$b(\sigma+1)+\delta_1^2+(\sigma/b)\delta_2\delta_3$	$b(\sigma+1)+\delta_1^2+(\sigma/b)\delta_R^2$	$b(\sigma+1)+\delta_1^2+(\sigma/b)Q^2$ $(Q^2=\delta_R^2+\omega^2)$	
$L_2 =$	$b(\sigma+1)+\delta_1^2+(\sigma/b)\delta_2\delta_3$	$b(\sigma+1)+\delta_R^2+(\sigma/b)\delta_1\delta_R$	——	$b(\sigma+1)+\delta^2(1+\sigma/b)\ (\frac{88}{3}+\frac{19}{4}\delta^2)$
$L_3 =$	$b(\sigma+1)+\delta_1^2+(\sigma/b)\delta_2\delta_3$	$(L_2 \equiv L_3 = L_R)$		
$M_1 =$	$\sigma[\delta_1^2+\delta_2\delta_3-\delta_1(\delta_2+\delta_3)]$	$\sigma(\delta_1-\delta_R)^2$	$\sigma[(\delta_1-\delta_R)^2+\omega^2]$	
$M_2 =$	$\sigma[\delta_2^2+\delta_3\delta_1-\delta_2(\delta_3+\delta_1)]$	$(M_2 \equiv M_3 = M_R \equiv 0)$		$M \equiv 0$
$M_3 =$	$\sigma[\delta_3^2+\delta_1\delta_2-\delta_3(\delta_1+\delta_2)]$		——	

Table 1. Stationary (δ_k) and stability (L, M) characteristics of the system (1.3).

2.2. Particular cases

We begin with the simplest case - one triple real root of (2.5), respectively (2.2). This is column D in the table. The only fixed point in this case is

$$P \equiv (\overline{X},\ \overline{Y},\ \overline{Z}) = (\delta,\ -2\delta,\ r-1+\delta^2/b) = (\delta,\ -2\delta, 27+3\delta^2/8). \tag{2.10}$$

Since $M = 0$ one eigenvalue is zero ($\lambda_1 = 0$). Hence $\lambda^2 + K\lambda + L(\delta) = 0$ and the other two are either negative or with $\lambda_R < 0$ independently of δ. Therefore, from the point of view of stability, this is a marginal case (Strogatz, 1994). However, numerical computations show that this point is stable.

Next in the Table 1 is the column C. The only real fixed point

$$P_1 \equiv (\overline{X}_1,\ \overline{Y}_1,\ \overline{Z}_1) = (\delta_1,\ -2\delta_R,\ r-1+Q^2/b),\ Q^2 = \delta_R^2+\omega^2 \tag{2.11}$$

is always stable. Indeed, as it is seen from the Table 1 $M_1 > 0$, while $KL - M > 0$ requires

$$\frac{3608}{9} + \frac{11}{3}\delta_1^2 + \frac{165}{4}Q^2 > -20\delta_1\delta_R \tag{2.12}$$

It is obvious that (2.12) is satisfied for $\delta_1\delta_R \geq 0$. In the opposite case ($\delta_1\delta_R < 0$), letting

$\delta_R = -\varepsilon\delta_1$, $\varepsilon > 0$ one transforms (2.12) into

$$\frac{3608}{9} + (\frac{11}{3} + \frac{165}{4}\varepsilon^3 - 20\varepsilon)\delta_1^2 + \frac{165}{4}\omega^2 > 0$$

However, if $\varepsilon > 0$ the coefficient in front of δ_1^2 never becomes negative, which means stable solution. Numerical computations confirm this conclusion.

Moving to column B we meet the case with two fixed points corresponding to δ_1 and $\delta_2 \equiv \delta_3 = \delta_R$:

$$P_1 \equiv (\overline{X}_1,\ \overline{Y}_2,\ \overline{Z}_1) = (\delta_1,\ -2\delta_R,\ r - 1 + \delta_R^2/b),$$

$$P_R \equiv (\overline{X}_R,\ \overline{Y}_R,\ \overline{Z}_R) = (\delta_R,\ -\delta_1 - \delta_R,\ r - 1 + \delta_1\delta_R/b). \tag{2.13}$$

The first one is a particular case of (2.11) at $\omega = 0$ and is always stable. More complicated is the situation with the second point R_R - one eigenvalue is always zero ($\lambda_1 = 0$), while

$$\lambda_{2,3} = -\frac{K}{2} \pm \sqrt{\frac{K^2}{4} - L_R}$$

where L_R is given in column B. Hence $\lambda_{2,3} < 0$ or $Re(\lambda_{2,3}) < 0$ at $\delta_1\delta_R > 0$. However, if $\delta_R = -\varepsilon\delta_1$, ($\varepsilon > 0$) then

$$L_R < 0 \quad \text{at} \quad 0 < \varepsilon < 15/4 \quad \text{and} \quad \delta_1^2 > \frac{88/3}{\varepsilon(\frac{15}{4} - \varepsilon)} = \delta_{cr}^2 \tag{1.13a}$$

so that $\lambda_2 \equiv \lambda_+ > 0$. Therefore, again we are dealing with a higher-order fixed point like (2.10). Numerical calculations confirm that expectancy for instability of the fixed point P_R under the conditions (2.13a).

Finally, the case of three unequal real fixed points (2.7) can be analized by the same way. This is the column A in the table. Each of the fixed points (2.7) has its own eigenvalue equation (2.3), i.e. its own coefficients L_K, M_K, but common $K = \sigma + b + 1 = 41/3$ and consequently its own stability properties.

3. The forced W-N system (1.6)

By equating $\dot{X}$, $\dot{Y}$, $\dot{Z}$ in (1.6) to zero we easily find

$$\overline{Y} = \pm\sqrt{\frac{m_3}{3m_2}(2m_1 + m_3)}, \qquad \overline{X} = \frac{m_3}{3\overline{Y}}, \qquad \overline{Z} = -\frac{1}{\overline{Y}}(m_1 + \frac{1}{3}m_3) = -\frac{m_1}{\overline{Y}} - \overline{X}. \tag{3.1}$$

Contrary to (1.11) only two fixed points exist. Since the system (1.6) is 3-dimensional, its characteristic equation will be cubic (2.3):

$$\lambda^3 - \overline{Y}\lambda^2 + \left[3\overline{Y}^3 + \frac{2}{3\overline{Y}^2}(m_3^2 - 2m_1m_3 - 3m_1^2)\right]\lambda - 6m_2\overline{Y} = 0 \tag{3.2}$$

so that in this case

$$K = -\overline{Y}, \qquad M = -6m_2\overline{Y},$$

$$L = 3\overline{Y}^2 + \frac{2}{3\overline{Y}^2}(m_3^2 - 2m_1m_3 - 3m_1^2) \tag{3.3}$$

By postulating (1.6) we have introduced the free parameters m_k which must satisfy the single condition

$$\frac{m_3}{3m_2}(2m_1 + m_3) = \delta^2 > 0, \quad \text{i.e.} \quad \overline{Y} = \pm\delta \tag{3.4}$$

where $\delta > 0$. Following the idea from the previous section 2, we shall consider δ as a primary free parameter to be chose at will. Furthermore, it is convenient to let

$$m_2 = \alpha m_1^2 \tag{3.5}$$

and to express m_3 by means of m_1, α, δ:

$$m_3 = m_1(-1 \pm \sqrt{q}), \qquad q = 1 + 3\alpha\delta^2. \tag{3.6}$$

According to the criteria (2.9), the fixed point with $\overline{Y} = \delta > 0$ is always unstable since $K = -\overline{Y} < 0$. For the second point $\overline{Y} = -\delta$ so that $K = \delta > 0$, $M = 6\alpha\delta m_1^2$. Hence, $\alpha > 0$ in order to have $M > 0$. Therefore, the stability of the second fixed point will depend on the third condition (2.9). The problem is still three-parametric (α, δ, m_1). We shall fix α and δ and consider m_1 as a bifurcation parameter. Under these assumptions and making use of (3.6) we obtain from (3.3)

$$L(m_1, \delta, q) = 3\delta^2 + \frac{2}{3\delta^2}(q \pm 4\sqrt{q})m_1^2. \tag{3.7}$$

We now check if the equaility $KL = M$ is possible under the obvious restrictions

$$m_1^2 > 0, \qquad \delta > 0, \qquad \alpha > 0 \qquad (q > 1). \tag{3.8}$$

If so, then the Hopf bifurcation will take place at $m_1 = m_H$. As a result one obtains

$$m_H^2 = \frac{(9/2)\delta^4}{2q \pm 4\sqrt{q} - 3} \tag{3.9}$$

where the signs "±" correspond to those in (3.6) and (3.7).

For the upper sign (+) the denominator in (3.9) is positive for any $q > 1$, i.e. $m_H^2 > 0$. However, for the lower sing (-)

$$m_H^2 > 0 \text{ at } q > (1 + \sqrt{5/2})^2 = 6.6623, \quad \text{or} \quad \alpha\delta^2 > 1.8874. \tag{3.10}$$

Having obtained these analytical results concerning the stationary and stability properties of the W-N system (1.6) one can continue its further numerical investigation. Particular cases can be selected for which $m_H^2 > 0$. For example, $\delta = 1$, i.e. $\overline{Y} = -1$ and

$$(m_1,\ m_2,\ m_3) = (m_1,\ m_1^2,\ m_1),\ (m_1,\ 5m_1^2,\ 3m_1) \text{ etc.} \tag{3.11}$$

as well as such values for which $m_H^2 < 0$, for example $m_1 = m_2 = m_3 > 0$. In all these cases the behaviour of the time dependent solution of (1.6) may be compared with the unforced system ($m_k = 0$). This work is left for a later time.

Here we present two preliminary examples only, corresponding to the upper sign in (3.6), at $\alpha = 1$, $\delta = 1$ ($q = 4$). Hence $(m_1,\ m_2,\ m_3) = (m_1,\ m_1^2,\ m_1)$, $(\overline{X},\ \overline{Y},\ \overline{Z}) = (-\frac{m_1}{3},\ -1,\ \frac{4}{3}m_1)$ and from (3.9) $m_H^2 = 9/26$. First we choose $m_1 = 0.2$ ($< m_H = 0.5883$). Calculations confirm the stability of the point (-0.66, -1, 0.254). However, at $m_1 = 0.7$ the behaviour of the system (1.6) around the fixed point is qualitatively different (Fig. 1). A cycle of period two is observed. Here and in Figure 2 a portion of the trajectory is shown as it spirals outward to the attractor from an initial condition (-0.231, -1, -0.889). Finally, at $m_1 = 0.73$ chaotic behaviour of $X(t)$ and a strange attractor projected on XY-plane is observed (Fig. 3).

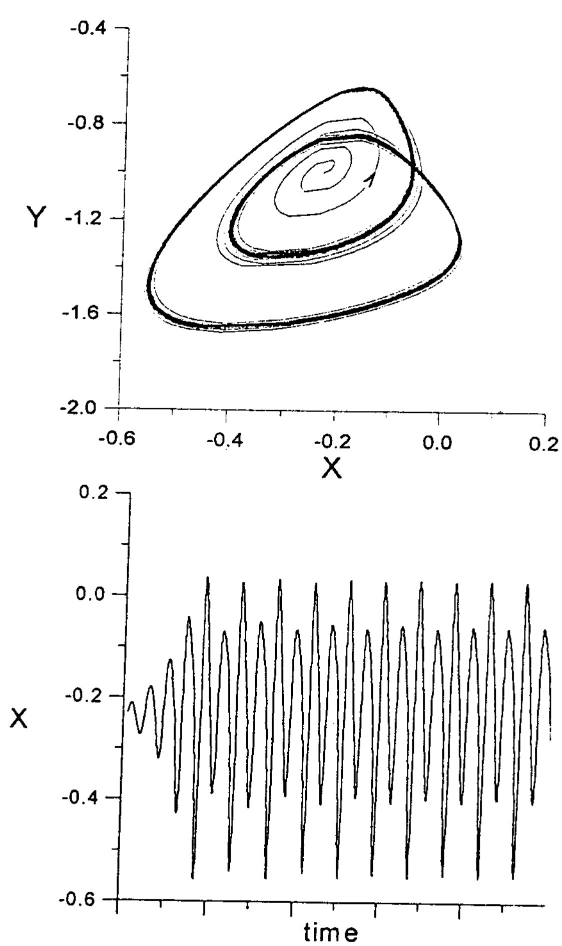

Fig. 1. A phase trajectory and $X(t)$ for $m_1 = 0.7$. The system has overgone the Hopf bifurcation and a cycle of period two is observed.

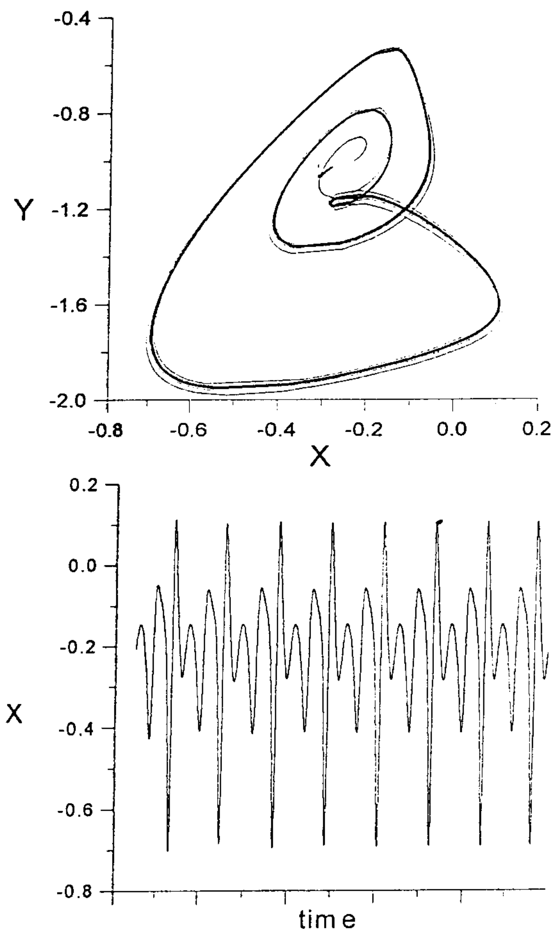

Fig. 2. The same as Figure 1 for $m_1 = 0.72$.

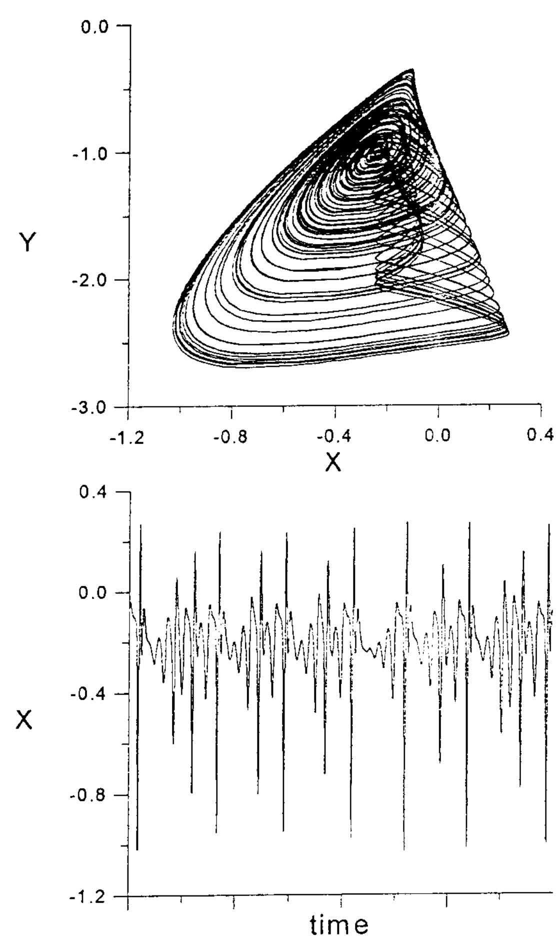

Fig. 3. Strange attractor and $X(t)$ observed at $m_1 = 0.73$.

4. Conclusion

The constant forcing of two known nonlinear dynamical systems is used as a tool for controlling the stationary and stability properties of these systems. Concerning the forced Lorenz system (1.3), some of the results are summarized on Table 1.

The homogeneous (unforced, $m_k = 0$) W-N system (1.6) cannot generate chaos because of the constraint (1.10). However, the forcing ($m_k \neq 0$) destroys (1.10) and opens possibilities for complex (periodic and chaotic) behaviour of the solution as illustrated in Figure 1 to Figure 3 for a particular example. In details the nuermical study of the W-N system as well as the L-system is left for later time.

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